

# Model checking for generalized linear models: a dimension-reduction model-adaptive approach

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**Abstract:** Local smoothing testing that is based on multivariate nonparametric regression estimation is one of the main model checking methodologies in the literature. However, relevant tests suffer from the typical curse of dimensionality resulting in slow convergence rates to their limits under the null hypotheses and less deviation from the null under alternatives. This problem leads tests to not well maintain the significance level and to be less sensitive to alternatives. In this paper, a dimension-reduction model-adaptive test is proposed for generalized linear models. The test behaves like a local smoothing test as if the model were univariate, and can be consistent against any global alternatives and can detect local alternatives distinct from the null at a fast rate that existing local smoothing tests can achieve only when the model is univariate. Simulations are carried out to examine the performance of our methodology. A real data analysis is conducted for illustration. The method can readily be extended to global smoothing methodology and other testing problems.

*Keywords:* Dimension reduction; generalized linear models; model-adaption; model checking.

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# 1 Introduction

Consider the following regression model:

$$Y = g(\beta^\tau \mathbf{X}, \theta) + \epsilon, \quad (1)$$

here  $Y$  is the scalar response,  $\mathbf{X}$  is a predictor vector of  $p$  dimension,  $g(\cdot)$  is a known squared integrable continuous function,  $\beta$  is any  $p$ -dimensional unknown parameter vector,  $\theta$  is an unknown parameter of  $d$ -dimension, and  $E(\epsilon|\mathbf{X}) = 0$ . Generalized linear model is one of its special cases.

To make statistical inference that is based on regression model reliable, we should carry out some suitable and efficient model checking procedures. There are various proposals in the literature for testing model (1) against a general alternative model:

$$Y = G(\mathbf{X}) + \epsilon, \quad (2)$$

here  $G(\cdot)$  is an unknown smooth function and  $E(\epsilon|\mathbf{X}) = 0$ . Two classes of methods are popularly used: the local smoothing methods and global smoothing methods. For the former type, see Härdle and Mammen (1993) in which the  $L_2$  distance between the null parametric regression and the alternative nonparametric regression was considered. Zheng (1996) proposed a quadratic form conditional moment test which was also independently developed by Fan and Li (1996). Fan et al. (2001) considered a generalized likelihood ratio test, and Dette (1999) developed a test that is based on the difference of two variance estimates under the null and alternative respectively. For other developments, see also the minimum distance test proposed by Koul and Ni (2004) and the distribution distance test developed by Van Keilegom et al. (2008). A relevant reference is Zhang and Dette (2004). For global smoothing methodologies that are based on empirical processes, the examples include the following methods. Stute (1997) introduced nonparametric principal component decomposition that is based on residual marked empirical process. Inspired by the Khmaladze transformation used in goodness-of-fitting for distributions, Stute et al. (1998b) first developed innovation martingale approach to obtain some distribution free tests. Stute and Zhu (2002) provided a relevant reference for generalized linear models. Khmaladze and Koul (2004) further studied the goodness-of-fit problem for errors in nonparametric regression. For a comprehensive review, see González-Manteiga and Crujeiras (2013).

Based on the simulation results in the literature, existing local smoothing methods are sensitive to high-frequency regression models and thus,

they often have high power to detect these alternative models. However, a very serious shortcoming is that these methods suffer seriously from the typical curse of dimensionality because of inevitable use of multivariate non-parametric function estimation. Specifically, it results in that existing local smoothing-based test statistics under null hypotheses converge to their limits at rates  $O(n^{-1/2}h^{-p/4})$  (or  $O(n^{-1}h^{-p/2})$  if the test is in a quadratic form) that are very slow when  $p$  is large. The readers can refer to Härdle and Mammen (1993) for a typical reference of this methodology. Further, the tests of this type can only detect alternatives distinct from the null hypothesis at the rates of order  $O(n^{-1/2}h^{-p/4})$  (see, e.g. Zheng 1996). This problem has been realized in the literature and there are a number of local smoothing tests that apply re-sampling or Monte Carlo approximation to help determine critical values (or  $p$  values). Relevant references include Härdle and Mammen (1993), Delgado and González-Manteiga (2001), Härdle et al. (2004), Dette et al. (2007), Neumeyer and Van Keilegom (2010). In contrast, though the rate is of order  $\sqrt{n}$ , most of existing global smoothing methods depend on high dimensional stochastic processes. See e.g. Stute et al. (1998a). Because of data sparseness in high-dimensional space, the power performance often drops significantly.

Therefore, it is of importance to consider how to make local smoothing methods get rid of the curse of dimensionality when model (1) is the hypothetical model that is actually of a dimension reduction structure. To motivate our method, we very briefly review the basic idea of existing local smoothing approaches. Under the null hypothesis,  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \mathbf{X}) = E(\epsilon | \mathbf{X}) = 0$  and under the alternative model (2),  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \mathbf{X}) \neq 0$ . Thus, its empirical version with root- $n$  consistent estimates of  $\beta$  and  $\theta$  is used as a base to construct test statistics. Its variant is that  $E(Y | \mathbf{X}) - g(\beta^\tau \mathbf{X}, \theta) = 0$ . The distance between a nonparametric estimate of  $E(Y | \mathbf{X})$  and a parametric estimate of  $g(\beta^\tau \mathbf{X}, \theta)$  is a base for test statistic construction (see, e.g. Härdle and Mammen 1993). The test that is based on  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \mathbf{X})$  can detect the alternative model (2). However, such a very natural idea inevitably involves high-dimensional nonparametric estimation of  $E(Y | \mathbf{X})$  or  $E(\epsilon | \mathbf{X})$ . This is the main cause of inefficiency in hypothesis testing with the rate of order  $O(n^{-1/2}h^{-p/4})$  as aforementioned.

To attack this problem, we note the following fact. Under the null hypothesis, it is clear that  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \mathbf{X}) = E(\epsilon | \beta^\tau \mathbf{X}) = 0$ . Thus, it leads to another naive idea to construct test statistic that is based on  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \beta^\tau \mathbf{X})$ . The test statistic construction sufficiently uses the information provided in the hypothetical model. From the technical development in the present paper, it is easy to see that a relevant test can have

the rate of order  $O(n^{-1/2}h^{-1/4})$  as if the dimension of  $\mathbf{X}$  were 1. A relevant reference is Stute and Zhu (2002) when global smoothing method was adopted. But this idea leads to another very obvious shortcoming that as test statistic is completely based on the hypothetical model, and thus it is actually a directional test rather than an omnibus test. It cannot handle the general alternative model (2). For instance, when the alternative model is  $E(Y|\mathbf{X}) = g(\beta^\tau \mathbf{X}, \theta) + g_1(\beta_1^\tau \mathbf{X})$  where  $\beta_1$  is orthogonal to  $\beta$  and  $\mathbf{X}$  follows the standard multivariate normal distribution  $N(0, I_p)$ . Then  $\beta^\tau \mathbf{X}$  is independent of  $\beta_1^\tau \mathbf{X}$ . When  $E(g_1(\beta_1^\tau \mathbf{X})) = 0$ , it is clear that, still, under this alternative model  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \beta^\tau \mathbf{X}) = 0$ . Thus, a test statistic that is based on  $E(Y - g(\beta^\tau \mathbf{X}, \theta) | \beta^\tau \mathbf{X})$  cannot detect the above alternative. Xia (2009) proposed a consistent test statistic by comparing the empirical cross-validation counterparts of the minimum of  $E^2(\epsilon - E(\epsilon | \alpha^\tau \mathbf{X}))$  over all unit vectors  $\alpha$  with the centered residual sum of squares. When the assumed model is adequate, the centered residual sum of squares should be small. However, this procedure cannot provide the corresponding limiting distributions under the null and alternatives and thus cannot test significance at a nominal level. As was pointed out by the author himself, under the null hypothesis, the rejection frequency tends to 0 as  $n \rightarrow \infty$ . In other words, this method cannot control the significance level in a large sample sense. Further, the computation is also an issue because cross-validation involves intensive computation. The author also provided a single-indexing bootstrap  $F$  test. But the consistency of this bootstrap method is not established. Thus, in certain sense it is hard for users to recognize type I and type II error by this test procedure.

Therefore, it is crucial for us to construct a test in the following way: using sufficiently the information under the null model (1) to avoid dimensionality problem as if the dimension of  $\mathbf{X}$  were 1, and adapting to the alternative model (2) such that the test can detect general alternatives. To achieve this goal, we will apply sufficient dimension reduction technique (SDR, Cook 1998). Define a more general alternative model than model (2). Note that  $\mathbf{X}$  can be rewritten as  $B^\tau \mathbf{X}$  where  $B = I_p$  an identity matrix. Model (2) can then be rewritten as  $E(Y|X) = \tilde{G}(B^\tau \mathbf{X})$ . It is worth noticing that this reformulation can be for any orthogonal  $p \times p$  matrix  $B$  because when  $G$  is an unknown function, the model can be rewritten as  $G(\mathbf{X}) = G(BB^\tau \mathbf{X}) := \tilde{G}(B^\tau \mathbf{X})$ . Thus,  $E(\epsilon | \mathbf{X}) = 0$  is equivalent to  $E(\epsilon | B^\tau \mathbf{X}) = 0$ , and  $E(G(\mathbf{X}) - g(\beta^\tau \mathbf{X}, \theta) | \mathbf{X}) \neq 0$  is equivalent to  $E(\tilde{G}(B^\tau \mathbf{X}) - g(\beta^\tau \mathbf{X}, \theta) | B^\tau \mathbf{X}) \neq 0$ . In other words, the model can be considered as a special multi-index model with  $p$  indexes. Based on this observation, we consider a more general alternative model that covers some

important models as special cases such as the single-index model and multi-index model:

$$Y = G(B^T \mathbf{X}) + \epsilon, \quad (3)$$

where  $B$  is a  $p \times q$  matrix with  $q$  orthogonal columns for an unknown number  $q$  with  $1 \leq q \leq p$  and  $G$  is an unknown smooth function. When  $q = p$ , this model is identical to model (2), and when  $q = 1$  and  $B = \beta/\|\beta\|$  is a column vector, model (3) reduces to a single-index model with the same index as that in the null model (1). Thus, it offers us a way to construct a test that is automatically adaptive to the null and alternative models through identifying and consistently estimating  $B$  (or  $BC$  for an  $q \times q$  orthogonal matrix) under both the null and alternatives. From this idea, a local smoothing test will be constructed in Section 2 that has two nice features under the null and alternatives: it sufficiently use the dimension reduction structure under the null and is still an omnibus test to detect general alternatives as existing local smoothing tests try to do. To be precise, the test statistic under the null converges to its limit at the faster rate of order  $O(n^{-1/2}h^{-1/4})$  (or  $O(n^{-1}h^{-1/2})$  if the test is in a quadratic form), is consistent against any global alternative and can detect local alternatives distinct from the null at the rate of order  $O(n^{-1/2}h^{-1/4})$ . This improvement is significant particularly when  $p$  is large because the new test does behave like a local smoothing test as if  $\mathbf{X}$  were one-dimensional. Thus, it is expectable that the test can well maintain the significance level and has better power performance than existing local smoothing tests.

The paper is organized by the following way. As sufficient dimension reduction (SDR, Cook 1998) plays a crucial role for identifying and estimating  $BC$  for an  $q \times q$  orthogonal matrix  $C$ , we will give a brief review for it in the next section. In Section 2, a dimension-reduction model-adaptive (DRMA) test is constructed. The asymptotic properties under the null and alternatives are investigated in Section 3. In Section 4, the simulation results are reported and a real data analysis is carried out for illustration. The basic idea can be readily used to other test procedures. For details, we leave this to Section 5. The proofs of the theoretical results are postponed to the appendix.

## 2 Dimension reduction model-adaptive test procedure

### 2.1 Basic test construction

Recall the hypotheses as: almost surely

$$H_0 : E(Y|\mathbf{X}) = g(\beta^\tau \mathbf{X}, \theta) \quad \text{versus} \quad H_1 : E(Y|\mathbf{X}) = G(B^\tau \mathbf{X}). \quad (4)$$

The null and alternative models can be reformulated as: under the null hypothesis,  $q = 1$  and then  $B = \tilde{\beta} = c\beta$  for some scalar  $c$  and under the alternative,  $q \geq 1$ . Thus,

$$E(\epsilon|\mathbf{X}) = 0 \implies E(\epsilon|\beta^\tau \mathbf{X}) = E(\epsilon|B^\tau \mathbf{X}) = 0.$$

Therefore, under  $H_0$ ,

$$E(\epsilon E(\epsilon|B^\tau \mathbf{X})W(B^\tau \mathbf{X})) = E(E^2(\epsilon|B^\tau \mathbf{X})W(B^\tau \mathbf{X})) = 0, \quad (5)$$

where  $W(X)$  is some positive weight function that is discussed below.

Under  $H_1$ , we have

$$E(\epsilon E(\epsilon|B^\tau \mathbf{X})W(B^\tau \mathbf{X})) = E(E^2(\epsilon|B^\tau \mathbf{X})W(B^\tau \mathbf{X})) > 0, \quad (6)$$

The empirical version of the left hand side in (5) can then be used as a test statistic, and  $H_0$  can be rejected for large values of the test statistic. To this end, we estimate  $E(\epsilon|B^\tau \mathbf{X})$  by, when a sample  $\{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}$  is available,

$$\hat{E}(\epsilon_i|\hat{B}(\hat{q})^\tau \mathbf{x}_i) = \frac{\frac{1}{n-1} \sum_{j \neq i}^n \hat{\epsilon}_j K_h(\hat{B}(\hat{q})^\tau \mathbf{x}_i - \hat{B}(\hat{q})^\tau \mathbf{x}_j)}{\frac{1}{n-1} \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau \mathbf{x}_i - \hat{B}(\hat{q})^\tau \mathbf{x}_j)}.$$

In this formula,  $\hat{\epsilon}_j = y_j - g(\hat{\beta}^\tau \mathbf{x}_j, \hat{\theta})$ ,  $\hat{\beta}$  and  $\hat{\theta}$  are the commonly used least squares estimate of  $\beta$  and  $\theta$ ,  $\hat{B}(\hat{q})$  is a sufficient dimension reduction estimate with an estimated structural dimension  $\hat{q}$  of  $q$ ,  $K_h(\cdot) = K(\cdot/h)/h^{\hat{q}}$  with  $K(\cdot)$  being a  $\hat{q}$ -dimensional kernel function and  $h$  being a bandwidth. As the estimations of  $B$  and  $q$  are crucial to the dimension reduction model-adaption test (DRMA), we will specify them later. When the weight  $W(\cdot)$  is chosen to be  $\hat{f}(\cdot)$  where  $\hat{f}(\hat{B}(\hat{q})^\tau \mathbf{X})$  is a kernel estimate of the density function  $f(\cdot)$  of  $B^\tau \mathbf{X}$  and, for any  $\hat{B}(\hat{q})^\tau \mathbf{x}_i$ ,

$$\hat{f}(\hat{B}(\hat{q})^\tau \mathbf{x}_i) = \frac{1}{n-1} \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau \mathbf{x}_i - \hat{B}(\hat{q})^\tau \mathbf{x}_j).$$

A non-standardized test statistic is defined by

$$V_n = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \hat{\epsilon}_i \hat{\epsilon}_j K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)). \quad (7)$$

**Remark 1.** *The test statistic suggested by Zheng (1996) is*

$$\tilde{V}_n = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \hat{\epsilon}_i \hat{\epsilon}_j \tilde{K}_h(\mathbf{x}_i - \mathbf{x}_j). \quad (8)$$

here  $\tilde{K}_h(\cdot) = K(\cdot/h)/h^p$  with  $K(\cdot/h)$  being a  $p$ -dimensional kernel function. Compared equation (7) with equation (8), there are two main differences. First, our test uses  $\hat{B}(\hat{q})^\tau \mathbf{X}$  in lieu of  $\mathbf{X}$  in Zheng (1996)'s test in the classical idea and applies  $K_h(\cdot)$  in  $V_n$  instead of  $\tilde{K}_h(\cdot)$ . This reduces the dimension  $p$  down to the dimension  $\hat{q}$ . Second, a further important ingredient in the test statistic construction is again about the use of kernel function. Under the null, we will show that  $\hat{q} \rightarrow 1$ , and  $\hat{B}(\hat{q}) \rightarrow c\beta$  for a constant  $c$ , and further  $nh^{1/2}V_n$  has finite limit. Under the alternative model (3), we will show that  $\hat{q} \rightarrow q \geq 1$  and  $\hat{B}(\hat{q}) \rightarrow BC$  for an  $q \times q$  orthogonal matrix. The estimation is then adaptive to the alternative model (3). This means that the test can be automatically adaptive to the underlying model, either the null or the alternative model.

## 2.2 Identification and estimation of $q$ and $B$

Note that  $B$  is a parameter matrix in the general regression model (3). In general,  $B$  is not identifiable because for any  $q \times q$  orthogonal matrix  $C$ ,  $G(B^\tau \mathbf{X})$  can also be written as  $\tilde{G}(C^\tau B^\tau \mathbf{X})$ . Again, similar to what was discussed before, it is enough to identify  $BC$  for an  $q \times q$  orthogonal matrix  $C$ . Thus, what we have to do is to identify  $BC$ . To achieve this, we will use the sufficient dimension reduction method (SDR, Cook 1998). Thus, we first briefly review SDR. Define the intersection of all subspaces  $S_B$  spanned by  $B$  for all  $p \times q$  matrices  $B$  such that  $Y \perp\!\!\!\perp E(Y|\mathbf{X})|B^\tau \mathbf{X}$  where  $\perp\!\!\!\perp$  means "independent of". Model (3) is a special case satisfying this conditional independence. This space, denoted  $S_{E(Y|\mathbf{X})}$ , is called the central mean subspace

(CMS, Cook and Li 2002). Thus, all we can identify is the subspace  $\mathcal{S}_{E(Y|\mathbf{X})}$  spanned by  $B$  rather than  $B$  itself in model (3). In other words, what we can identify is  $\tilde{B} = BC$  for a  $q \times q$  orthogonal matrix or equivalently,  $q$  base vectors of  $\mathcal{S}_{E(Y|\mathbf{X})}$ .  $q$  is called the structural dimension of the central mean subspace  $\mathcal{S}_{E(Y|\mathbf{X})}$ . There are several proposals in the literature. The examples include sliced inverse regression (SIR, Li 1991), sliced average variance estimation (SAVE, Cook and Weisberg 1991), contour regression (CR, Li et al. 2005), directional regression (DR, Li and Wang 2007), likelihood acquired directions (LAD, Cook and Forzani 2009), discretization-expectation estimation (DEE, Zhu et al. 2010a) and average partial mean estimation (APME, Zhu et al. 2010b). In addition, minimum average variance estimation (MAVE, Xia et al, 2002) can identify and estimate the relevant space with fewer regularity conditions on  $\mathbf{X}$ , but requires nonparametric smoothing on involved nonparametric regression function. As DEE and MAVE have good performance in general, we review these two methods below.

### 2.3 A review on discretization-expectation estimation

As described in above, our test procedure needs to estimate the matrix  $B$ . In this subsection, we first assume the the dimension  $q$  is known in ahead and then we discuss how to select the dimension  $q$  consistently. We first give a brief review of discretization-expectation estimation (DEE), see Zhu et al. (2010a) for details. In sufficient dimension reduction, SIR and SAVE are two popular methods which involve the partition of the range of  $Y$  into several slices and the choice of the number of slices. However, as documented by many authors, for instance, Li (1991), Zhu and Ng (1995) and Li and Zhu (2007), the choice of the number of slices may effect the efficiency and can even yield inconsistent estimates. To avoid the delicate choice of the number of slices, Zhu et al. (2010a) introduced the discretization-expectation estimation (DEE). The basic idea is simple. We first define the new response variable  $Z(t) = I(Y \leq t)$ , which takes the value 1 if  $Y \leq t$  and 0 otherwise. Let  $\mathcal{S}_{Z(t)|\mathbf{X}}$  be the central subspace and  $\mathcal{M}(t)$  be a  $p \times p$  positive semi-definite matrix such that  $\text{span}\{\mathcal{M}(t)\} = \mathcal{S}_{Z(t)|\mathbf{X}}$ . Define  $\mathcal{M} = E\{\mathcal{M}(T)\}$ . Under certain mild conditions, we can have  $\mathcal{M}$  is equal to the central subspace that contains the central mean subspace  $\mathcal{S}_{E(Y|\mathbf{X})}$ . For details, readers can refer to Zhu et al. (2010).

In the discretization step, we construct a new sample  $\{\mathbf{x}_i, z_i(y_j)\}$  with  $z_i(y_j) = I(y_i \leq y_j)$ . For each fixed  $y_j$ , we estimate  $\mathcal{M}(y_j)$  by using SIR or SAVE. Let  $\mathcal{M}_n(y_j)$  denote the candidate matrix obtained from a chosen



method such as SIR. In the expectation step, we can estimate  $\mathcal{M}$  by  $\mathcal{M}_{n,n} = n^{-1} \sum_{j=1}^n \mathcal{M}_n(y_j)$ . The  $q$  eigenvectors of  $\mathcal{M}_{n,n}$  corresponding to its  $q$  largest eigenvalues can be used to form an estimate of  $B$ . Denote the DEE procedure based on SIR and SAVE be  $DEE_{SIR}$  and  $DEE_{SAVE}$  respectively. To save space, in this paper, we only focus on these two basic methods. The resulting estimate  $\hat{B}(q)$  can be defined by  $DEE_{SIR}$  or  $DEE_{SAVE}$ . Zhu et al. (2010a) proved that  $\hat{B}(q)$  is consistent to  $BC$  for a  $q \times q$  non-singular matrix  $C$  and the given  $q$ .

## 2.4 A review on minimum average variance estimation

As well known, SIR and SAVE needs the linearity and constant conditional variance conditions for us to successfully estimate  $\mathcal{S}_{E(Y|\mathbf{X})}$  (Li, 1991, Cook and Weisberg 1991). In contrast, the minimum average conditional variance Estimation (MAVE) requires fewer regularity conditions, while asks local smoothing in high-dimensional space. In the following, we also apply MAVE to estimate the base vectors of this subspace, see Xia et al. (2002) for details. The estimate  $\hat{B}$  is the minimizer of

$$\sum_{j=1}^n \sum_{i=1}^n (y_i - a_j - \mathbf{d}_j^T B^T \mathbf{x}_{ij})^2 K_h(B^T \mathbf{x}_{ij}),$$

over all  $B$  satisfying  $B^T B = I_q$ ,  $a_j$  and  $\mathbf{d}_j$ , here  $\mathbf{d}_j = G'(B^T \mathbf{x}_j)$  and  $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ . The details of the algorithm can be referred to Xia et al. (2002). The resulting estimate  $\hat{B}(q)$  is also consistent to  $BC$  for an  $q \times q$  orthogonal matrix when  $q$  is given. When it is unknown, an estimate of  $q$  is involved, which is stated below.

## 2.5 Estimation of the structural dimension $q$

Consider the DEE-based and MAVE-based estimates. According to Zhu et al. (2010a), we determine  $q = \dim(\mathcal{S}_{Y|\mathbf{X}})$  by

$$\hat{q} = \arg \max_{l=1, \dots, p} \left\{ \frac{n}{2} \times \frac{\sum_{i=1}^l \{\log(\hat{\lambda}_i + 1) - \hat{\lambda}_i\}}{\sum_{i=1}^p \{\log(\hat{\lambda}_i + 1) - \hat{\lambda}_i\}} - 2 \times D_n \times \frac{l(l+1)}{2p} \right\},$$

where  $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \dots \geq \hat{\lambda}_p \geq 0$  are the eigenvalues of  $\mathcal{M}_{n,n}$ , and  $D_n$  is a constant to be determined by user. We should note that the first term in the bracket can be considered as likelihood ratio, and the second term is the

penalty term with  $l(l+1)/2$  free parameters when the dimension is  $l$ . Zhu et al. (2010a) explained the calculation of this number of free parameters in details. See also Zhu et al. (2006) for more discussions of this methodology of BIC type. The major merit of this methodology is that the consistency of  $\hat{q}$  only requires the convergence of  $\mathcal{M}_{n,n}$ . Zhu et al. (2010a) proved that under some regularity conditions,  $\hat{q}$  is a consistent estimate of  $q$ . Following their suggestion, we choose  $D_n = n^{1/2}$ .

For MAVE, we instead suggest a BIC criterion that is a modified version of that proposed by Wang and Yin (2008), which has the following form:

$$BIC_k = \log\left(\frac{RSS_k}{n}\right) + \frac{\log(n)k}{\min(nh^k, \sqrt{n})},$$

where  $RSS_k$  is the residual sum of squares, and  $k$  is the estimate of the dimension. The form of  $RSS_k$  is as follows:

$$RSS_k = \sum_{j=1}^n \sum_{i=1}^n (y_i - \hat{a}_j - \hat{\mathbf{d}}_j^\tau \hat{B}(k)^\tau \mathbf{x}_{ij})^2 K_h(\hat{B}(k)^\tau \mathbf{x}_{ij}),$$

here we use  $B(k)$  to denote the matrix  $B$  when the dimension is  $k$ .

The estimated dimension is then

$$\hat{q} = \min\{l : l = \arg \min_{1 \leq k \leq p} \{BIC_k\}\}.$$

Wang and Yin (2008) showed that under some mild conditions,  $\hat{q}$  is also a consistent estimate of  $q$ .

**Proposition 1.** *Under the conditions assumed in Zhu et al. (2010a), the DEE-based estimate  $\hat{q}$  is consistent to  $q$ . Also under the conditions assumed in Wang and Yin (2008), the MAVE-based estimate  $\hat{q}$  is consistent to  $q$ . Therefore, the estimate  $\hat{B}(\hat{q})$  is a consistent estimate of  $BC$  for a  $q \times q$  orthogonal matrix  $C$ .*

It is worth pointing out that these consistencies are under the null and global alternative. Under the local alternatives that will be specified in Section 3, the results are different, we will show the consistency of  $\hat{q}$  to 1.

### 3 Asymptotic properties

#### 3.1 Limiting null distribution

Give some notations first. Let  $\mathbf{Z} = \beta^\tau \mathbf{X}$ ,  $\sigma^2(\mathbf{z}) = E(\epsilon^2 | \mathbf{Z} = \mathbf{z})$ , and

$$\begin{aligned} Var &= 2 \int K^2(u) du \cdot \int (\sigma^2(\mathbf{z}))^2 f^2(\mathbf{z}) d\mathbf{z}, \\ \widehat{Var} &= \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h^{\hat{q}}} K^2\left(\frac{\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)}{h}\right) \hat{\epsilon}_i^2 \hat{\epsilon}_j^2. \end{aligned}$$

We will show that  $\widehat{Var}$  is consistent to  $Var$  under the null and local alternatives. This estimation will make the standardized test more sensitive to the alternative. We now state the asymptotic property of the test statistic under the null hypothesis.

**Theorem 1.** *Under  $H_0$  and the conditions in the Appendix, we have*

$$nh^{1/2}V_n \Rightarrow N(0, Var),$$

*Further,  $Var$  can be consistently estimated by  $\widehat{Var}$ .*

We now standardize  $V_n$  to get a scale-invariant statistic. According to Theorem 1, the standardized  $V_n$  is

$$\begin{aligned} T_n &= \sqrt{\frac{n-1}{n}} \frac{nh^{1/2}V_n}{\sqrt{\widehat{Var}}} \\ &= \frac{h^{(1-\hat{q})/2} \sum_{i=1}^n \sum_{j \neq i}^n \hat{\epsilon}_i \hat{\epsilon}_j K\left(\frac{\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)}{h}\right)}{\{2 \sum_{i=1}^n \sum_{j \neq i}^n K^2\left(\frac{\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)}{h}\right) \hat{\epsilon}_i^2 \hat{\epsilon}_j^2\}^{1/2}}. \end{aligned}$$

By the consistency of  $\widehat{Var}$ , the application of the Slutsky theorem yields the following corollary.

**Corollary 1.** *Under the conditions in Theorem 1 and  $H_0$ , we have*

$$T_n^2 \Rightarrow \chi_1^2,$$

*where  $\chi_1^2$  is the chi-square distribution with one degree of freedom.*

From this corollary, we can then calculate  $p$  values easily by its limiting null distribution. As a popularly used approach, Monte Carlo simulation can also be used. We will discuss this in the simulation studies in Section 4.

### 3.2 Power study

We now examine the power performance of the proposed test statistic under a sequence of alternatives with the form

$$H_{1n} : Y = g(\beta^\tau \mathbf{X}, \theta) + C_n G(B^\tau \mathbf{X}) + \eta, \quad (9)$$

where  $E(\eta|\mathbf{X}) = 0$  and the function  $G(\cdot)$  satisfies  $E(G^2(B^\tau \mathbf{X})) < \infty$ . When  $C_n$  is a fixed constant, it is under global alternative, whereas when  $C_n \rightarrow 0$ , the models are local alternatives. In this sequence of models,  $\beta$  is one of the columns in  $B$ .

Denote  $m(\mathbf{X}, \beta, \theta) = \text{grad}_{\beta, \theta}(g(\beta^\tau \mathbf{X}, \theta))^\tau$ ,  $H(\mathbf{X}) = G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)$  and  $\Sigma_x = E(m(\mathbf{X}, \beta, \theta)m(\mathbf{X}, \beta, \theta)^\tau)$ .

Before presenting the main results about the power performance under the alternative (9), we give the results about the estimator  $\hat{q}$  of  $q$  under the local alternatives. This is because the local alternative converges to the null model and thus, it is expected that  $\hat{q}$  would also tend to 1.

**Theorem 2.** *Under the local alternatives of (9), when the conditions in Appendix hold, we have that  $\hat{q} \rightarrow 1$ . Here  $\hat{q}$  is either the DEE-based estimate or the MAVE-based estimate.*

Now, we are ready to present the power performance.

**Theorem 3.** *Under the conditions in Appendix, we have the following.*

(i) *Under the global alternative of 3)*

$$T_n/(nh^{1/2}) \Rightarrow \text{Constant} > 0.$$

(ii) *Under the local alternatives of (9),  $C_n = n^{-1/2}h^{-1/4}$ ,  $nh^{1/2}V_n \Rightarrow N(\mu, \text{Var})$*

*and  $T_n^2 \Rightarrow \chi_1^2(\mu^2/\text{Var})$ , where*

$$\mu = E \left[ \left( G(B^\tau \mathbf{X}) - m(\mathbf{X}, \beta, \theta)^\tau \Sigma_x^{-1} E[H(\mathbf{X})] \right)^2 f(\beta^\tau \mathbf{X}) \right],$$

here  $\chi_1^2(\mu^2/\text{Var})$  is a noncentral chi-squared random variable with one degree of freedom and the noncentrality parameter  $\mu^2/\text{Var}$ .

**Remark 2.** *This theorem reveals two important phenomena when compared with Zheng’s test: the new test tends to infinity at the rate of order  $nh^{1/2}$  whereas Zheng’s test goes to infinity at the rate of order  $nh^{p/2}$  a much slower rate to infinity; the new test can detect the local alternatives distinct from the null at the rate of order  $C_n = n^{-1/2}h^{-1/4}$ , whereas the optimal rate for detectable local alternatives the classical tests can achieve is  $C_n = n^{-1/2}h^{-p/4}$ , see e.g. Härdle and Mammen (1993) and Zheng (1996). These two facts show the very significant improvement of the new test achieved.*

## 4 Numerical analysis

### 4.1 Simulations

We now carry out simulations to examine the performance of the proposed test. Because the situation is similar, we then consider linear models instead of generalized linear models as the hypothetical models in the following studies. Further to save space, we only consider the SIR-based DEE procedure and MAVE in the following. In this section, we consider 3 simulation studies. In Study 1, the matrix  $B$  is  $\beta$  under both the null and alternative. Thus, we use this study to examine the performance for the models with this simple structure and compare the performance when DEE and MAVE are used to determine  $BC$  for an  $q \times q$  orthogonal matrix. Study 2 is to check, through a comparison with Stute and Zhu’s test (2002), how our test is an omnibus test that has the advantage to detect general alternative models rather than a directional test that cannot do this job. Study 3 is multi-purpose: to check the impact from the dimensionality for both our test and a local smoothing test (Zheng 1996). It is noted that we choose Zheng’s test for comparison because 1). it has tractable limiting null distribution and we can see its performance when limiting null distribution is used to determine the critical values (or  $p$  values); 2). Like other local smoothing tests, we can also use its

re-sampling version to determine the critical values (or  $p$  values) to examine its performance. Thus, we use it as a representative of local smoothing tests to make comparison. We also conduct a comparison with Härdle and Mammen's (1993) test in a small scale simulation, the conclusions are very similar and thus the results are not reported in the present paper.

*Study 1.* Consider

$$\begin{aligned} H_{11} : Y &= \beta^\tau \mathbf{X} + a \cos(0.6\pi\beta^\tau \mathbf{X}) + \epsilon; \\ H_{12} : Y &= \beta^\tau \mathbf{X} + a \exp\{-(\beta^\tau \mathbf{X})^2\} + \epsilon; \\ H_{13} : Y &= \beta^\tau \mathbf{X} + a(\beta^\tau \mathbf{X})^2 + \epsilon. \end{aligned}$$

The value  $a = 0$  corresponds to the null hypothesis and  $a \neq 0$  to the alternatives. In other words, even under the alternative, the model is still single-indexed. In the simulation,  $\beta = (1, 1, \dots, 1)^\tau / \sqrt{p}$ ,  $\mathbf{X} = (X_1, X_2, \dots, X_p)^\tau$  and  $p$  is set to be 8. The observations  $\mathbf{x}_i, i = 1, 2, \dots, n$ , are i.i.d. respectively from multivariate normal distribution  $N(0, \Sigma_j), j = 1, 2$ , with

$$\Sigma_1 = I_{p \times p}; \quad \Sigma_2 = (0.5^{|j-l|})_{p \times p}.$$

Further,  $\epsilon$  follows the standard normal distribution  $N(0, 1)$ . In this simulation study, the replication time is 2,000.

In the nonparametric regression estimation, the kernel function is taken to be  $K(u) = 15/16(1 - u^2)^2$ , if  $|u| \leq 1$ ; and 0 otherwise. The bandwidth is taken to be  $h = 1.5n^{-1/(4+\hat{q})}$  with separately standardized predictors for simplicity. We have this recommended value because to investigate the impact of bandwidth selection, we tried different bandwidth  $h$  to be  $n^{-1/(4+\hat{q})}(0.25 + i/4)$  for  $i = 0, \dots, 8$ . For the three models, we found similar pattern we depict in Figure 1 that is under  $H_{11}$  with  $X \sim N(0, \Sigma_1)$ , and the sample size 50 at the nominal level  $\alpha = 0.05$ . From this figure, we can see that the test is somehow robust to the bandwidth selection when empirical size is a concern: with different bandwidths, our proposed test can control the size well. On the other hand, the bandwidth selection does have impact for power performance especially when the bandwidth is too small. Overall, from our empirical experience here,  $h = 1.5n^{-1/(4+\hat{q})}$  is recommendable.

Figure 1 about here

Now we turn to study the empirical sizes and powers of our proposed test against alternatives  $H_{1i}, i = 1, 2, 3$  with the nominal size  $\alpha = 0.05$ . The results are shown in Tables 1–3. Write respectively  $T_n^{DEE}$  and  $T_n^{MAVE}$  as

the corresponding test statistic  $T_n$  that is respectively based on DEE and MAVE. From these three tables, we can have the following observations. In all the cases we consider,  $T_n^{DEE}$  controls the size very well even under the small sample size  $n = 50$ . This suggests that we can rely on the limiting null distribution to determine critical values (or  $p$  values).

For  $T_n^{MAVE}$ , our simulations which are not reported here show that the empirical sizes intend to be slightly larger than 0.05. Thus an adjustment on the test statistic is needed. The following size-adjustment would be recommendable:

$$\tilde{T}_n^{MAVE} = \frac{T_n^{MAVE}}{1 + 4n^{-4/5}}.$$

The test statistic is asymptotically equivalent to  $T_n^{MAVE}$ , but has better performance in small or moderate sample size. The results shown in Tables 1–3 are with this adjusted test statistic.

As was pointed out in the literature and commented in the previous sections, local smoothing tests usually suffer from dimensionality problem such that they cannot work well in the significance level maintenance with good power performance. Thus, re-sampling techniques are often employed in finite sample paradigms. A typical technique is the wild bootstrap first suggested by Wu (1986) and well developed later (see, e.g. Härdle and Mamenn 1993). Consider the bootstrap observations:

$$y_i^* = \hat{\beta}^\tau \mathbf{x}_i + \hat{\epsilon}_i \times V_i.$$

Here  $\{V_i\}_{i=1}^n$  is a sequence of i.i.d. random variables with zero mean, unit variance and independent of the sequence  $\{y_i, \mathbf{x}_i\}_{i=1}^n$ . Usually,  $\{V_i\}_{i=1}^n$  can be chosen to be i.i.d. Bernoulli variates with

$$P(V_i = \frac{1 - \sqrt{5}}{2}) = \frac{1 + \sqrt{5}}{2\sqrt{5}}, \quad P(V_i = \frac{1 + \sqrt{5}}{2}) = 1 - \frac{1 + \sqrt{5}}{2\sqrt{5}}.$$

Let  $T_n^*$  be defined similarly as  $T_n$ , basing on the bootstrap sample  $(\mathbf{x}_1, y_1^*), \dots, (\mathbf{x}_n, y_n^*)$ . The null hypothesis is rejected if  $T_n$  is bigger than the corresponding quantile of the bootstrap distribution of  $T_n^*$ . The bootstrap versions of  $T_n$  based on DEE and MAVE are respectively written as  $T_n^{DEE*}$  and  $T_n^{MAVE*}$ .

From Tables 1–3, we can see the following. The empirical sizes of  $T_n^{MAVE*}$  are slightly larger than 0.05 especially when  $\mathbf{X} \sim N(0, \Sigma_1)$ .  $\tilde{T}_n^{MAVE}$  can control the size acceptably in different situations.  $T_n^{DEE*}$  can also maintain the type I error very well, but cannot work better than  $T_n^{DEE}$ . In summary,  $T_n^{DEE}$ ,  $T_n^{DEE*}$ ,  $T_n^{MAVE*}$  and  $\tilde{T}_n^{MAVE}$  all work well. Thus, re-sampling

technique seems not necessary for our DRMA tests though  $T_n^{MAVE}$  needs some adjustment to get  $\tilde{T}_n^{MAVE}$ . For power performance, Tables 1–3 suggest that when  $\mathbf{X} \sim N(0, \Sigma_1)$ ,  $\tilde{T}_n^{MAVE}$  generally has higher power than  $T_n^{DEE}$  has. Yet, when  $\mathbf{X}$  follows  $N(0, \Sigma_2)$ ,  $T_n^{DEE}$  becomes the winner. Further,  $T_n^{MAVE*}$  has slightly higher power than  $T_n^{DEE*}$  has. Moreover, for the alternatives  $H_{11}$  and  $H_{12}$ ,  $T_n^{MAVE*}$  generally has relatively higher power than  $\tilde{T}_n^{MAVE}$  has while for the alternative  $H_{13}$ ,  $\tilde{T}_n^{MAVE}$  is more powerful than  $T_n^{MAVE*}$ . As for the DEE-based tests, in almost all cases,  $T_n^{DEE}$  can have higher power than its bootstrapped version  $T_n^{DEE*}$ . These tests are all very sensitive to the alternatives. To be precise, when  $a$  increases, the powers can increase very quickly. It seems that the MAVE-based tests tend to be more conservative than the DEE-based tests, and  $T_n^{DEE}$  can not only control the size satisfactorily but also have high power in the limited simulations.

Tables 1–3 about here

Further, as commented in Section 1, the test proposed by Stute and Zhu (2002) is also a dimension reduction test that is however a directional test. Specifically, they developed an innovation process transformation of the empirical process  $n^{-1/2} \sum_{i=1}^n \left( y_i - g(\hat{\beta}^\tau \mathbf{x}_i, \hat{\theta}) \right) I(\hat{\beta}^\tau \mathbf{x}_i \leq u)$ . By introducing the transformation, the test is asymptotically distribution-free, but not an omnibus test though it has been proved to be powerful in many scenarios (see, e.g. Stute and Zhu 2002; Mora and Moro-Egido 2008). We use the following simulation to demonstrate this claim in the finite sample cases.

*Study 2.* The data are generated from the following model:

$$Y = \beta_1^\tau \mathbf{X} + a(\beta_2^\tau \mathbf{X})^3 + \epsilon. \quad (10)$$

Consider two cases of  $\beta_i$ . The first is  $\beta_1 = (1, 0, 0)^\tau$ ,  $\beta_2 = (0, 1, 0)^\tau$  for  $p = 3$ ; the second one is  $\beta_1 = (1, 1, 0, 0)^\tau / \sqrt{2}$ ,  $\beta_2 = (0, 0, 1, 1)^\tau / \sqrt{2}$  for  $p = 4$ . When  $p = 3$ , the sample size  $n = 50, 100$ , and when  $p = 4$ ,  $n = 100$ . In both the cases,  $X$  and  $\epsilon$  are generated from multivariate and univariate standard normal distribution. Further, consider  $a = 0.0, 0.3, \dots, 1.5$  to examine power performance. Write Stute and Zhu (2002)'s test as  $T_n^{SZ}$ . To save space, we only present the results of  $T_n^{DEE}$  in Figure 2. It is obvious that  $T_n^{DEE}$  uniformly performs much better than  $T_n^{SZ}$ .  $T_n^{SZ}$  can have very low powers. In contrast,  $T_n^{DEE}$  can efficiently detect the alternatives. Here, we only use this simulation to show  $T_n^{SZ}$  is a directional test rather than to show  $T_n^{SZ}$  is a bad test. Actually, it has been proved to be a good test in



other scenarios (see Stute and Zhu 2002). Further, for other comparisons, it will be a comparison between local smoothing tests and global smoothing tests. We will have further investigation in an ongoing research.

Figure 2 about here

To evidence the performance of our test when there are more than one direction under the alternative hypothesis and the impact from dimensionality, we construct the following simulation.

*Study 3.* The data are generated from the following model:

$$Y = \beta_1^\tau \mathbf{X} + a(\beta_2^\tau \mathbf{X})^2 + \epsilon, \quad (11)$$

where  $\beta_1 = (\underbrace{1, \dots, 1}_{p/2}, 0, \dots, 0)^\tau / \sqrt{p/2}$ ,  $\beta_2 = (0, \dots, 0, \underbrace{1, \dots, 1}_{p/2})^\tau / \sqrt{p/2}$ .

Thus, under the null, we have  $B = \beta_1$  and under the alternatives,  $B = (\beta_1, \beta_2)$ . In this study, we consider  $p = 2$  and 8 to examine how the dimension affects Zheng's test and ours.

The observations  $\mathbf{x}_i, i = 1, 2, \dots, n$  are generated from multivariate normal distribution  $N(0, \Sigma_j), j = 1, 2$  and  $\epsilon$  from  $N(0, 1)$  and the double exponential distribution  $DE(0, \sqrt{2}/2)$  with density  $f(x) = \sqrt{2}/2 \exp\{-\sqrt{2}|x|\}$  with mean zero and variance 1 respectively. To save space, we only consider  $T_n^{DEE}$  and  $T_n^{DEE*}$  in the following due to their well performance on size control and easy computation.

When  $p = 2$ ,  $\beta_1 = (1, 0)^\tau$  and  $\beta_2 = (0, 1)^\tau$ . The results are reported in Table 4. From this table, firstly, we can observe that Zheng (1996)'s test  $T_n^{ZH}$  can maintain the significance level reasonably in some cases, but usually, lower than it. When the bootstrap is used,  $T_n^{ZH*}$  performs better in this aspect. In contrast, both  $T_n^{DEE}$  and  $T_n^{DEE*}$  can maintain the significance level very well. In other words, the DRMA test we developed does not need the assistance from the bootstrap approach, while Zheng's test is eager for. For the empirical powers, we can find that generally,  $T_n^{ZH}$  and  $T_n^{DEE}$  have higher powers than their bootstrap version. However, for our tests, the differences are negligible. This again shows that the bootstrap helps little for our test. Whereas Zheng (1996)'s test also needs the help from the bootstrap for power performance. Further, our tests, both  $T_n^{DEE}$  and  $T_n^{DEE*}$  are uniformly and significantly more powerful than Zheng (1996)'s test and the bootstrap version.

Table 4 about here

We now consider the  $p = 8$  case. The results are reported in Table 5. It is clear that the dimension very significantly deteriorates the performance of Zheng's test. Table 5 indicates that the empirical size of  $T_n^{ZH}$  is far away from the significance level. The bootstrap can help on increasing the empirical size, but still not very close to the level. In contrast,  $T_n^{DEE}$  can again maintain the significance level very well and the bootstrap version  $T_n^{DEE*}$  does not show its advantage over  $T_n^{DEE}$ . Further, the power performance of  $T_n^{ZH}$  becomes much worse than that under the  $p = 2$  case shown in Table 4, even the bootstrap version does not enhance the power, which is much lower than that of  $T_n^{DEE}$ . This further shows that  $T_n^{ZH}$  is affected by the dimension badly, while  $T_n^{DEE}$  is not. The empirical powers of  $T_n^{DEE}$  and  $T_n^{DEE*}$  with  $p = 8$  are even higher than those with  $p = 2$ . This dimensionality blessing phenomenon is of interest and worthy of a further exploration in another research.

Table 5 about here

These findings coincide with the theoretical results derived before: existing local smoothing tests have much slower convergence rate (of order  $n^{-1/2}h^{-p/4}$  or  $n^{-1}h^{-p/2}$  if the test is a quadratic form) to their limit under the null and less sensitive to local alternative (at the rate of order  $n^{-1/2}h^{-p/4}$  to the null) than the DRMA test we developed. The simulations we conducted above shows that the DRMA test can simply use its limiting null distribution to determine critical values without heavy computational burden, and has high power.

## 4.2 Real data analysis

This dataset is obtained from the Machine Learning Repository at the University of California-Irvine (<http://archive.ics.uci.edu/ml/datasets/Auto+MPG>). Recently, Xia (2007) analysed this data set by their method. The first analysis of this data set is due to Quinlan (1993). There are 406 observations in the original data set. To illustrate our method, we first clear the units with missing response and/or predictor and get 392 sample points. The response variable  $Y$  is miles per gallon ( $Y$ ). There are other seven predictors: the number of cylinders ( $X_1$ ), engine displacement ( $X_2$ ), horsepower ( $X_3$ ), vehicle weight ( $X_4$ ), time to accelerate from 0 to 60 mph ( $X_5$ ), model year ( $X_6$ ) and origin of the car (1 = American, 2 = European, 3 = Japanese). Since the origin of the car contains more than two categories, we follow

Xia (2007)'s suggestions and define two indicator variables. To be precise, let  $X_7 = 1$  if a car is from America and 0 otherwise and  $X_8 = 1$  if it is from Europe and 0 otherwise. For ease of explanation, all the predictors are standardized separately. Quinlan (1993) aimed to predict the response in terms of the eight predictors  $\mathbf{X} = (X_1, \dots, X_8)^T$ . To achieve this goal, a simple linear regression model was adopted. However, as was argued in Introduction, we need to check its adequacy to avoid model misspecification. The value of  $T_n^{DEE}$  is 86.5703 and the p value is 0. The value of  $\hat{T}_n^{MAVE}$  is 98.2602 and the p value is also 0. Hence linear regression model is not plausible to predict the response. Figure 3 suggests that a nonlinear model should be used. Moreover, the  $\hat{q}$  is estimated to be 1 by the criterion with DEE in the paper. Thus a single-index model may be applied.

Figure 3 about here

## 5 Discussions

In this paper, we propose a dimension-reduction model-adaptive test procedure and use Zheng's test as an example to construct test statistic. It is readily extended to other local smoothing methods discussed in Section 1. The same principle can be applied to global smoothing methods. To be precise, as discussed in Section 2, under the null hypothesis  $Y = g(\beta^T \mathbf{X}, \theta) + \epsilon$  with  $E(\epsilon|\mathbf{X}) = 0$ , we can have  $E(\epsilon|\beta^T \mathbf{X}) = E(\epsilon|B^T \mathbf{X}) = 0$ . While under the alternative  $H_1$ ,  $E(\epsilon|B^T \mathbf{X}) = E(Y - g(\beta^T \mathbf{X}, \theta)|B^T \mathbf{X}) = G(B^T \mathbf{X}) - g(\beta^T \mathbf{X}, \theta) \neq 0$ . This motivates us to define the following test statistics:

$$R_n(\mathbf{z}) = n^{-1/2} \sum_{i=1}^n \left( y_i - g(\hat{\beta}^T \mathbf{x}_i, \hat{\theta}) \right) I(\hat{B}(\hat{q})^T \mathbf{x}_i \leq \mathbf{z}).$$

This is different from that in Stute and Zhu (2002) in which  $\hat{B}(\hat{q}) = \hat{\beta}$  in  $R_n(\mathbf{z})$  for generalized linear models. As we commented and compared in theoretical development and simulations, Stute and Zhu's (2002) test is a directional test and then is inconsistent under general alternatives. An ongoing project is doing for this.

Further, extensions of our methodology to missing, censored data and dependent data set can also be considered. Take the missing response as an example. Let  $\delta_i$  be the missing indicator, that is,  $\delta_i = 1$  if  $y_i$  is observed, otherwise it's equal to zero. Assume that the response is missing at random. This means  $P(\delta = 1|\mathbf{X}, Y) = P(\delta = 1|\mathbf{X}) := \pi(\mathbf{X})$ . For more details,

see Little and Rubin(1987). Again, we consider to test whether the following regression model holds or not. Namely,  $H_0 : Y = g(\beta^\tau \mathbf{X}, \theta) + \epsilon$  with  $E(\epsilon|\mathbf{X}) = 0$  and  $Y$  is missing at random. Note that under the null hypothesis  $E(\delta\epsilon/\pi(\mathbf{X})|\beta^\tau \mathbf{X}) = E(\delta\epsilon/\pi(\mathbf{X})|B^\tau \mathbf{X}) = 0$ , while under the alternative  $E(\delta\epsilon/\pi(\mathbf{X})|B^\tau \mathbf{X}) \neq 0$ . Similarly, we can construct a consistent test statistic with the following form:

$$V_{n1} = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{\delta_i}{\hat{\pi}(\mathbf{x}_i)} \frac{\delta_j}{\hat{\pi}(\mathbf{x}_j)} \hat{\epsilon}_i \hat{\epsilon}_j K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)),$$

here  $\hat{\pi}(\mathbf{x}_i)$  is an estimate, say, the nonparametric or parametric estimate, of  $\pi(\mathbf{x}_i)$ ,  $\hat{\epsilon}_i = y_i - g(\hat{\beta}^\tau \mathbf{x}_i, \hat{\theta})$  and  $\hat{\beta}$  and  $\hat{B}(\hat{q})$  is obtained from the complete observed units. Another possible test statistic takes the following form

$$V_{n2} = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \delta_i \delta_j \hat{\epsilon}_i \hat{\epsilon}_j K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)).$$

This corresponds to the test statistics obtained from the complete case.

Also we can consider applying the methodology to other testing problems such as testing for homoscedasticity, testing for parametric quantile regression model and testing for conditional parametric density function of  $Y$  given  $\mathbf{X}$ . The relevant research is ongoing.

In summary, the methodology is an general method that can be readily applied to many testing problems.

## Appendix. Proof of the theorems

### 5.1 Conditions

The following conditions are assumed for the theorems in Section 3.

- 1)  $\sup E(X_l^2|B^\tau \mathbf{X}) < \infty, l = 1, \dots, p; E(\eta^2|B^\tau \mathbf{X}) < \infty, \sup G^2(B^\tau \mathbf{X}) < \infty$  and there exists an integrable function  $L(x)$  such that  $|m_i(\mathbf{X}, \beta, \theta)| \leq L(x)$  for all  $(\beta, \theta)$  and  $1 \leq i \leq d + p$ .  $g(\mathbf{X}^\tau \beta, \theta)$  is a Borel measurable function on  $R^p$  for each  $\beta, \theta$  and a twice continuously differentiable real function on a compact subset of  $R^p$  and  $R^d$ ,  $\Lambda$  and  $\Theta$  for each  $x \in R^p$ ;
- 2)  $nh^2 \rightarrow \infty$  under the null (1) and local alternative hypothesis (9);  $nh^q \rightarrow \infty$  under global alternative hypothesis (3).

- 3) The density  $f(B^\tau \mathbf{X})$  of  $B^\tau \mathbf{X}$  on support  $\mathcal{C}$  exists and has 2 bounded derivatives and satisfies

$$0 < \inf_{B^\tau \mathbf{X} \in \mathcal{C}} f(B^\tau \mathbf{X}) \leq \sup_{B^\tau \mathbf{X} \in \mathcal{C}} f(B^\tau \mathbf{X}) < \infty;$$

- 4)  $K(\cdot)$  is a spherically symmetric density function with a bounded derivative and support, and all the moments of  $K(\cdot)$  exist and  $\int U U^\top K(U) dU = I$ .
- 5) Let  $\gamma = (\beta, \theta)$  and  $\tilde{\gamma}_0$ , the value of  $\gamma$  that minimizes  $\tilde{S}_{0n}(\gamma) = E[(E(Y|X) - g(\mathbf{X}^\tau \beta, \theta))^2]$ , is an interior point and is the unique minimizer of the function  $\tilde{S}_{0n}$ .  $\Sigma_x = E(m(\mathbf{X}, \beta, \theta)m(\mathbf{X}, \beta, \theta)^\tau)$  is positive definite.

**Remark 3.** Condition 1) is necessary for the root- $n$  consistency of the least squares estimates  $\hat{\beta}$  and  $\hat{\theta}$ . Condition 2) is needed for the asymptotic normality of our statistic. In Condition 2),  $nh^2 \rightarrow \infty$  is an usual assumption in nonparametric estimation. Conditions 3) and 4) are commonly used in nonparametric estimation. Condition 5) is necessary for the asymptotic normality of relevant estimators.

## 5.2 Lemmas

**Lemma 1.** Let  $\mathcal{M}(t)$  be a  $p \times p$  positive semi-definite matrix such that  $\text{span}\{\mathcal{M}(t)\} = \mathcal{S}_{Z(t)|\mathbf{X}}$ . We can have  $\text{span}\{E\{\mathcal{M}(T)\}\} = \text{span}\{E\{\mathcal{M}(T)\rho(T)\}\}$ , here  $\rho(\cdot) > 0$  is some weight function.

*Proof of Lemma 1.* Let  $\mathbf{v} \perp \text{span}\{E\{\mathcal{M}(T)\}\}$ , we can have  $0 = E\{\mathbf{v}^\tau \mathcal{M}(T) \mathbf{v}\}$ . Due to the fact that  $\mathcal{M}(t)$  is semi-definite matrix for any  $t$ , we can obtain that  $\mathcal{M}(t)\rho(t)\mathbf{v} = 0$ . In other words,  $\mathbf{v} \perp \text{span}\{\mathcal{M}(t)\rho(t)\}$  for any  $t$ . Further, we can get  $E\{\mathcal{M}(T)\rho(T)\mathbf{v}\} = 0$ . Thus  $\mathbf{v} \perp \text{span}\{E\{\mathcal{M}(T)\rho(T)\}\}$ . This follows that  $\text{span}\{E\{\mathcal{M}(T)\rho(T)\}\} \subseteq \text{span}\{E\{\mathcal{M}(T)\}\}$ . Another direction can be similarly shown. We conclude that  $\text{span}\{E\{\mathcal{M}(T)\}\} = \text{span}\{E\{\mathcal{M}(T)\rho(T)\}\}$ .

**Lemma 2.** *Under the null hypothesis and conditions 1)-4), we have*

$$W_n = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)) \epsilon_i M(x_{jl}) = O_p(1/\sqrt{n}), l = 1, \dots, p.$$

where  $M(\cdot)$  is continuously differentiable and  $E(M^2(X_l)|B^\tau \mathbf{X}) \leq b(B^\tau \mathbf{X})$  for  $X_l \in R$  and  $E[b(B^\tau \mathbf{X})] < \infty$ .

*Proof of Lemma 2.* Under null hypothesis,  $B = c\beta$  and  $\hat{B}(\hat{q})$  is consistent to  $B$ . For notational convenience, denote  $B_{ij} = B^\tau(\mathbf{x}_i - \mathbf{x}_j)$  and  $\hat{B}(\hat{q})_{ij} = \hat{B}(\hat{q})^\tau(\mathbf{x}_i - \mathbf{x}_j)$ . Further, define  $\tilde{W}_n$  as

$$\tilde{W}_n = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} K\left(\frac{\hat{B}(\hat{q})_{ij}}{h}\right) \epsilon_i M(x_{jl}) = \frac{h^{\hat{q}}}{h} W_n.$$

Since  $\hat{q} \rightarrow 1$  in probability, we only need to show  $\tilde{W}_n = O_p(1/\sqrt{n})$ . Note that

$$\begin{aligned} \tilde{W}_n &= \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} K\left(\frac{B_{ij}}{h}\right) \epsilon_i M(x_{jl}) \\ &\quad + \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} \left( K\left(\frac{\hat{B}(\hat{q})_{ij}}{h}\right) - K\left(\frac{B_{ij}}{h}\right) \right) \epsilon_i M(x_{jl}) \\ &= W_{n1} + W_{n2}. \end{aligned}$$

Let  $\mathbf{t}_i = (y_i, \mathbf{x}_i^\tau)^\tau$ , then  $W_{n1}$  can be written in a U-statistic with the kernel as

$$H_n(\mathbf{t}_i, \mathbf{t}_j) = \frac{1}{2h} K\left(\frac{B_{ij}}{h}\right) [\epsilon_i M(x_{jl}) + \epsilon_j M(x_{il})].$$

To apply the theory for non-degenerate U-statistic (Serfling 1980), we need

to show  $E[H_n^2(\mathbf{t}_i, \mathbf{t}_j)] = o(n)$ . Let  $\mathbf{Z} = B^\tau \mathbf{X}$ . It can be verified that

$$\begin{aligned}
& E[H_n^2(\mathbf{t}_i, \mathbf{t}_j)] \\
& \leq 2E \left[ \frac{1}{2h} K\left(\frac{B_{ij}}{h}\right) \epsilon_i M(x_{jl}) \right]^2 + 2E \left[ \frac{1}{2h} K\left(\frac{B_{ij}}{h}\right) \epsilon_j M(x_{il}) \right]^2 \\
& = \int \frac{1}{h^2} \sigma^2(z_i) E(M^2(x_{jl})|z_j) K^2\left(\frac{z_i - z_j}{h}\right) f(z_i) f(z_j) dz_i dz_j \\
& \leq \int \frac{1}{h} \sigma^2(z_i) b(z_i - hu) K^2(u) f(z_i) f(z_i - hu) dz_i du \\
& = \int \frac{1}{h} \sigma^2(z) b(z) f^2(z) dz \cdot \int K^2(u) du + o(1/h) \\
& = O(1/h) = o(n).
\end{aligned}$$

Since  $E(\epsilon|\mathbf{X}) = 0$ , it can be derived that  $E(H_n(\mathbf{t}_i, \mathbf{t}_j)) = 0$ . Now, consider the conditional expectation of  $H_n(\mathbf{t}_i, \mathbf{t}_j)$ . Also, it is easy to compute that

$$\begin{aligned}
r_n(\mathbf{t}_i) &= E(H_n(\mathbf{t}_i, \mathbf{t}_j)|\mathbf{t}_i) = \frac{\epsilon_i}{2h} E \left( K\left(\frac{B^\tau(\mathbf{x}_i - \mathbf{X})}{h}\right) M(X_l) \right) \\
&= \frac{\epsilon_i}{2h} E \left( K\left(\frac{z_i - Z}{h}\right) E(M(X_l)|Z) \right) = \frac{\epsilon_i}{2} \int E(M(X_l)|z_i + hu) f(z_i + hu) K(u) du \\
&= \frac{\epsilon_i f(z_i) E(M(X_l)|z_i)}{2} + l_n(\mathbf{t}_i).
\end{aligned}$$

Denote  $\hat{W}_n$  as the “projection” of the statistic  $W_{n1}$  as:

$$\begin{aligned}
\sqrt{n} \hat{W}_n &= \frac{2}{\sqrt{n}} \sum_{i=1}^n r_n(\mathbf{t}_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i f(z_i) E(M(X_l)|z_i) + \frac{2}{\sqrt{n}} \sum_{i=1}^n l_n(\mathbf{t}_i) \\
&= O_p(1).
\end{aligned}$$

The last equation follows from the fact that  $E(l_n^2(\mathbf{t}_i)) = O(h^2) \rightarrow 0$  due to the Lipschitz condition for the function  $E(M(X_l)|\cdot)f(\cdot)$ . As a result, we have  $W_{n1} = O_p(\hat{W}_n) = O_p(1/\sqrt{n})$ . Denote

$$W_{n2}^* = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} K'\left(\frac{B_{ij}}{h}\right) (\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i M(X_{jl}) \times \frac{\hat{B}(\hat{q}) - B}{h}.$$

Then for the term  $W_{n2}$ , we have

$$W_{n2} = W_{n2}^* + o_p(W_{n2}^*).$$

Since  $K(\cdot)$  is spherically symmetric, similar to  $W_{n1}$ , the following term

$$\frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} K'(\frac{B_{ij}}{h})(\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i M(X_{ji})$$

can be rewritten as a U-statistic. Then we can similarly show that this term is also of order  $O_p(1/\sqrt{n})$ . As  $\|\hat{B}(\hat{q}) - B\|_2 = O_p(1/\sqrt{n})$ , and under the condition  $1/nh^2 \rightarrow 0$ , we can obtain that  $W_{n2} = o_p(1/\sqrt{n})$ . Thus we can conclude that  $W_n = O_p(1/\sqrt{n})$ . The proof is completed.  $\square$

Before we establish the asymptotic theory of our statistic under the null and local alternatives, we develop the following lemma about the asymptotic property of  $\hat{\beta}$  and  $\hat{\theta}$ . This is necessary because it is defined under the null hypothesis.

**Lemma 3.** *Under the local alternative and conditions 1), 5), we have*

$$\begin{aligned} \sqrt{n}(\hat{\gamma} - \gamma) &= \Sigma_x^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) \eta_i + \Sigma_x^{-1} C_n \sqrt{n} E(m(\mathbf{X}, \beta, \theta) G(B^\tau \mathbf{X})) \\ &\quad + (\tilde{\Sigma}^{-1} - \Sigma_x^{-1}) C_n \sqrt{n} E(m(\mathbf{X}, \beta, \theta) G(B^\tau \mathbf{X})) + o_p(1). \end{aligned}$$

Here  $\tilde{\Sigma} = n^{-1} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) m(\mathbf{x}_i, \beta, \theta)^\tau$  and  $\gamma = (\beta, \theta)^\tau$ .

*Proof of Lemma 3.* Under the regularity conditions designed in Jennrich (1969), similar to the derivation of Theorems 6 and 7 in Jennrich (1969),  $\hat{\gamma}$  is a strongly consistent estimate of  $\gamma$ . If we let  $E_n = \tilde{\Sigma}^{-1} n^{-1} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta)(y_i -$



$g(\beta^\tau \mathbf{x}_i, \theta)$ ), we can further have:

$$\begin{aligned}
\hat{\gamma} - \gamma &= \tilde{\Sigma}^{-1} \frac{1}{n} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) (y_i - g(\beta^\tau \mathbf{x}_i, \theta)) + o_p(E_n) \\
&= (\tilde{\Sigma}^{-1} - \Sigma_x^{-1}) \frac{1}{n} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) \eta_i \\
&\quad + (\tilde{\Sigma}^{-1} - \Sigma_x^{-1}) \frac{1}{n} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) C_n G(B^\tau \mathbf{x}_i) \\
&\quad + \Sigma_x^{-1} \frac{1}{n} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) \eta_i \\
&\quad + \Sigma_x^{-1} \frac{1}{n} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) C_n G(B^\tau \mathbf{x}_i) + o_p(E_n) \\
&=: \sum_{i=1}^4 I_{ni} + o_p(E_n). \tag{12}
\end{aligned}$$

Due to the consistency of  $\tilde{\Sigma}$  for  $\Sigma_x$ , we can easily conclude that

$$\begin{aligned}
\sqrt{n}(\hat{\gamma} - \gamma) &= \Sigma_x^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{x}_i, \beta, \theta) \eta_i + \Sigma_x^{-1} C_n \sqrt{n} E(m(\mathbf{X}, \beta, \theta) G(B^\tau \mathbf{X})) \\
&\quad + (\tilde{\Sigma}^{-1} - \Sigma_x^{-1}) C_n \sqrt{n} E(m(\mathbf{X}, \beta, \theta) G(B^\tau \mathbf{X})) + o_p(1).
\end{aligned}$$

### 5.3 Proofs of the theorems

*Proof of Theorem 1.* First,  $V_n$  can be decomposed as, noting the symmetry of  $K_h(\cdot)$ ,

$$\begin{aligned}
V_n &= \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})_{ij}) \epsilon_i \epsilon_j \\
&\quad - \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})_{ij}) \epsilon_i m(\mathbf{x}_j, \beta, \theta)^\tau (\hat{\gamma} - \gamma) \\
&\quad + (\hat{\gamma} - \gamma)^\tau \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})_{ij}) m(\mathbf{x}_i, \beta, \theta) m(\mathbf{x}_j, \beta, \theta)^\tau (\hat{\gamma} - \gamma) \\
&\quad + o_p(V_n^*) \\
&=: V_{n1} - V_{n2} + V_{n3} + o_p(V_n^*), \tag{13}
\end{aligned}$$

where  $V_n^*$  denotes the term  $V_{n1} - V_{n2} + V_{n3}$ .

Consider the term  $V_{n2}$ . Under the conditions designed for Theorem 1, and from Lemmas 2 and 3, we can get that  $V_{n2} = O_p(1/n)$ . This yields that  $nh^{1/2}V_{n2} = o_p(1)$ .

Now we deal with the term  $V_{n3}$ . Rewrite it as

$$V_{n3} = (\hat{\gamma} - \gamma)^\tau \cdot \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})_{ij}) m(\mathbf{x}_i, \beta, \theta) m(\mathbf{x}_j, \beta, \theta)^\tau \cdot (\hat{\gamma} - \gamma).$$

Under null hypothesis,  $B = c\beta$  and  $\hat{B}(\hat{q})$  is consistent to  $B$ . A similar argument for proving Lemma 2 can be used to derive that

$$\begin{aligned} & \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})_{ij}) m(\mathbf{x}_i, \beta, \theta) m(\mathbf{x}_j, \beta, \theta)^\tau \\ &= E(m(\mathbf{X}, \beta, \theta) m(\mathbf{X}, \beta, \theta)^\tau f(\beta^\tau \mathbf{X})) + o_p(1). \end{aligned}$$

By the rate of  $\hat{\gamma} - \gamma$ ,  $V_{n3} = O_p(1/n)$ . Consequently,  $nh^{1/2}V_{n3} = o_p(1)$ .

Finally, deal with the term  $V_{n1}$ . Note that we have the following decomposition:

$$\begin{aligned} V_{n1} &= \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(B_{ij}) \epsilon_i \epsilon_j \\ &\quad + \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n (K_h(\hat{B}(\hat{q})_{ij}) - K_h(B_{ij})) \epsilon_i \epsilon_j \\ &=: V_{n1,1} + V_{n1,2}. \end{aligned}$$

For the term  $V_{n1,1}$ , since in this paper, we always assume that the dimension of  $B^\tau \mathbf{X}$  is fixed, it is an U-statistic. Note that under the null hypothesis,  $q = 1$  and  $\hat{q} \rightarrow 1$ . It is easy to derive the asymptotic normality:  $nh^{1/2}V_{n1,1} \rightarrow N(0, Var)$ . Here

$$Var = 2 \int K^2(u) du \cdot \int (\sigma^2(\mathbf{z}))^2 f^2(\mathbf{z}) d\mathbf{z}$$

with  $\mathbf{Z} = B^\tau \mathbf{X}$ ,  $\sigma^2(\mathbf{z}) = E(\epsilon^2 | \mathbf{Z} = \mathbf{z})$ . See a similar argument as that for Lemma 3.3 a) in Zheng (1996). We then omit the details.

Denote

$$V_{n1,2}^* = \frac{h}{h^{\hat{q}}} \cdot \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} K'(\frac{B_{ij}}{h}) (\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i \epsilon_j \cdot \frac{\hat{B}(\hat{q}) - B}{h}.$$

An application of Taylor expansion yields

$$V_{n1,2} = V_{n1,2}^* + o_p(V_{n1,2}^*).$$

Since  $K(\cdot)$  is spherical symmetric, the term

$$\frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h} K'(\frac{B_{ij}}{h})(\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i \epsilon_j$$

can be considered as an U-statistic. Further note that

$$\begin{aligned} & E \left( K'(\frac{B_{ij}}{h})(\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i \epsilon_j | \mathbf{x}_i, y_i \right) \\ &= E \left( E \{ K'(\frac{B_{ij}}{h})(\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i \epsilon_j | \mathbf{x}_i, y_i, \mathbf{x}_j \} | \mathbf{x}_i, y_i \right) \\ &= E \left( K'(\frac{B_{ij}}{h})(\mathbf{x}_i - \mathbf{x}_j)^\tau \epsilon_i E \{ \epsilon_j | \mathbf{x}_j \} | \mathbf{x}_i, y_i \right) = 0. \end{aligned}$$

Thus the above U-statistic is degenerate. Using a similar argument for the term  $V_{n1,1}$  above, together with  $\|\hat{B}(\hat{q}) - B\|_2 = O_p(1/\sqrt{n})$  and  $1/nh^2 \rightarrow 0$ , it results in that  $nh^{1/2}V_{n1,2}^* = o_p(1)$ . Thus we can have  $nh^{1/2}V_{n1} \rightarrow N(0, Var)$ .

Combining the above results for the terms  $V_{ni}, i = 1, 2, 3$ , we conclude that

$$nh^{1/2}V_n \Rightarrow N(0, Var).$$

Since  $Var$  is actually unknown, an estimate is defined as

$$\widehat{Var} = \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h^{\hat{q}}} K^2(\frac{\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)}{h}) \epsilon_i^2 \epsilon_j^2.$$

As the proof is rather straightforward, we then only give a very brief description. Since  $\hat{\beta}$  is consistent under the null hypothesis, some elementary computations lead to an asymptotic presentation:

$$\widehat{Var} = \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h^{\hat{q}}} K^2(\frac{\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)}{h}) \epsilon_i^2 \epsilon_j^2 + o_p(1).$$

Using a similar argument as that for Lemma 2, we get

$$\widehat{Var} = \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{h^{\hat{q}}} K^2(\frac{B^\tau (\mathbf{x}_i - \mathbf{x}_j)}{h}) \epsilon_i^2 \epsilon_j^2 + o_p(1).$$

The consistency will be derived by using U-statistic theory. The proof is finished.  $\square$

*Proof of Theorem 2.* We first investigate the  $\hat{q}$  determined by the BIC criterion for DEE. We also adopt the same conditions as those in Theorem 4 of Zhu et al. (2010a).

From the argument used for proving Theorem 3.2 of Li et al. (2008), we can know that to obtain that  $\mathcal{M}_{n,n} - \mathcal{M} = O_p(C_n)$ , we only need to show that  $\mathcal{M}_n(t) - \mathcal{M}(t) = O_p(C_n)$  uniformly. Now we investigate the term  $\mathcal{M}_n(t)$  for every  $t$ . Here  $\mathcal{M}(t) = \Sigma^{-1} \text{Var}(E(\mathbf{X}|Z(t))) = \Sigma^{-1}(\mu_1 - \mu_0)(\mu_1 - \mu_0)^\tau p(1-p)$  here  $\Sigma$  is the covariance matrix of  $\mathbf{X}$ ,  $\mu_j = E(\mathbf{X}|Z(t) = j)$  with  $j = 0$  and  $1$  and  $p = E(I(Y \leq t))$ . Further note that

$$\begin{aligned} \mu_1 - \mu_0 &= \frac{E(\mathbf{X}I(Y \leq t))}{p} - \frac{E(\mathbf{X}I(Y > t))}{1-p} \\ &= \frac{E(\mathbf{X}I(Y \leq t)) - E(\mathbf{X})E(I(Y \leq t))}{p(1-p)}. \end{aligned}$$

Thus from Lemma 1,  $\mathcal{M}(t)$  can also be taken to be

$$\mathcal{M}(t) = \Sigma^{-1} \left[ E\{(\mathbf{X} - E(\mathbf{X}))I(Y \leq t)\} \right] \left[ E\{(\mathbf{X} - E(\mathbf{X}))I(Y \leq t)\} \right]^\tau.$$

For ease of illustration, we denote  $m(t) = E\{(\mathbf{X} - E(\mathbf{X}))I(Y \leq t)\}$ , then  $\mathcal{M}(t) = \Sigma^{-1}m(t)m(t)^\tau = \Sigma^{-1}L(t)$ .  $m(t)$  is estimated by

$$m_n(t) =: n^{-1} \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})I(y_i \leq t).$$

Thus,  $\mathcal{M}_n(t)$  can be taken to be

$$\mathcal{M}_n(t) = \hat{\Sigma}^{-1}m_n(t)m_n^\tau(t) = \hat{\Sigma}^{-1}L_n(t).$$

Denote the response under the null and local alternative as  $Y$  and  $Y_n$  respectively. Note that

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i I(y_{in} \leq t) - E(\mathbf{X}I(Y \leq t)) \\ &= \frac{1}{n} \sum_{i=1}^n [\mathbf{x}_i I(y_{in} \leq t) - E(\mathbf{X}I(Y_n \leq t))] + E(\mathbf{X}I(Y_n \leq t)) - E(\mathbf{X}I(Y \leq t)). \end{aligned}$$

By the Lindeberg-Levy central limit theorem, the first term has the rate of order  $O_p(n^{-1/2})$ . Now we consider the second term. Note that

$$E(\mathbf{X}I(Y_n \leq t)) - E(\mathbf{X}I(Y \leq t)) = E\left(\mathbf{X}[P(Y_n \leq t|\mathbf{X}) - P(Y \leq t|\mathbf{X})]\right).$$

Recall that  $Y_n = Y + C_n G(B^\tau \mathbf{X})$  and denote the conditional density and distribution function of  $Y$  given  $\mathbf{X}$  as  $f_{Y|\mathbf{X}}(\cdot)$  and  $F_{Y|\mathbf{X}}(\cdot)$  respectively. We can then have

$$\begin{aligned} P(Y_n \leq t|\mathbf{X}) - P(Y \leq t|\mathbf{X}) &= F_{Y|\mathbf{X}}(t - C_n G(B^\tau \mathbf{X})) - F_{Y|\mathbf{X}}(t) \\ &= -C_n G(B^\tau \mathbf{X}) f_{Y|\mathbf{X}}(t) + O_p(C_n). \end{aligned}$$

From this, we can conclude that  $n^{-1} \sum_{i=1}^n \mathbf{x}_i I(y_{in} \leq t) - E(\mathbf{X}I(Y \leq t)) = O_p(C_n)$ . Similarly,  $m_n(t) - m(t) = O_p(C_n)$ ,  $L_n(t) - L(t) = O_p(C_n)$  and  $\mathcal{M}_n(t) - \mathcal{M}(t) = O_p(C_n)$ . Finally, similar to the argument used for proving Theorem 3.2 of Li et al. (2008),  $\mathcal{M}_{n,n} - \mathcal{M} = O_p(C_n)$ . Now we turn to prove the consistency of BIC criterion-based estimate when DEE is used. By the definition in Section 2, for  $l > 1$ , we have

$$G(1) - G(l) = D_n \frac{l(l+1) - 2}{p} - \frac{n \sum_{i=2}^l \log(\hat{\lambda}_i + 1) - \hat{\lambda}_i}{2 \sum_{i=1}^p \log(\hat{\lambda}_i + 1) - \hat{\lambda}_i}.$$

Invoking  $\mathcal{M}_{n,n} - \mathcal{M} = O_p(C_n)$ ,  $\hat{\lambda}_i - \lambda_i = O_p(C_n)$ . Note that  $\log(\hat{\lambda}_i + 1) - \hat{\lambda}_i = -\hat{\lambda}_i^2 + o_p(\hat{\lambda}_i^2)$  and  $\lambda_i = 0$  for any  $i > 1$ . We can obtain that  $\sum_{i=2}^l \log(\hat{\lambda}_i + 1) - \hat{\lambda}_i = O_p(C_n^2)$  and  $\sum_{i=1}^p \log(\hat{\lambda}_i + 1) - \hat{\lambda}_i \rightarrow b$  in probability for some  $b < 0$ . Taking  $D_n = n^{1/2}$  and  $C_n = n^{-1/2}h^{-1/4}$ , it is easy to see that

$$\frac{nC_n^2}{D_n} = (nh)^{-1/2} \rightarrow 0.$$

Since  $l(l+1) > 2$  for any  $l > 1$ ,  $P(G(1) > G(l)) \rightarrow 1$ . In other words,  $P(\hat{q} = 1) \rightarrow 1$ .

From Zhu et al. (2010a), the matrix  $\mathcal{M}$  that is based on either SIR or SAVE can satisfy the consistency  $\mathcal{M}_{n,n}$  requires. Thus, the estimate  $\hat{q}$  that is based on  $DEE_{SIR}$  or  $DEE_{SAVE}$  can have the above consistency. We then omit the detail here.

We now turn to consider the  $\hat{q}$  being selected by the BIC criterion with MAVE. Assume the same conditions in Wang and Yin (2008), and

for simplicity assume that  $\mathbf{X}$  has a compact support over which its density is positive. Recall the definition of  $B(k)$  for any  $k$ , we have that  $Y = E(Y|B^\tau(k)\mathbf{X}) + \epsilon$ , where  $E(\epsilon|B^\tau(k)\mathbf{X}) = 0$ . Suppose that the orthogonal  $p \times k$  matrix  $\hat{B}(k)$  is the sample estimate of  $B(k)$ .

Note that under the local alternative hypothesis (9), we can have

$$\begin{aligned} Y &= g(\beta^\tau \mathbf{X}, \theta) + C_n G(B^\tau \mathbf{X}) + \eta; \\ E(Y|B^\tau(k)\mathbf{X}) &= g(\beta^\tau \mathbf{X}, \theta) + C_n E(G(B^\tau \mathbf{X})|B^\tau(k)\mathbf{X}). \end{aligned}$$

Similar to the argument used in Wang and Yin (2008), under the local alternatives, we have

$$\begin{aligned} \frac{1}{n}RSS_k &= E\{Y - E(Y|B^\tau(k)\mathbf{X})\}^2 + O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{nh^k}\right) \\ &= E\{\eta^2\} + O_p(C_n^2) + O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{nh^k}\right). \end{aligned}$$

The second equation is based on the fact that  $E\{\eta[G(B^\tau \mathbf{X}) - E(G(B^\tau \mathbf{X})|B^\tau(k)\mathbf{X})]\} = 0$  by using  $E(\eta|\mathbf{X}) = 0$ . Consequently, the following results hold:

$$\frac{RSS_k - RSS_1}{n} = O_p(C_n^2) + O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{nh^k}\right).$$

Recall that the BIC criterion is  $BIC_k = \log(RSS_k/n) + \log(n)k / \min(nh^k, \sqrt{n})$ . Thus we can obtain that

$$BIC_k - BIC_1 = \log(RSS_k/RSS_1) + \frac{\log(n)k}{\min(nh_{k,opt}^k, \sqrt{n})} - \frac{\log(n)}{\min(nh_{1,opt}, \sqrt{n})}.$$

Consider the term  $\log(RSS_k/RSS_1)$  first. We can easily get

$$\begin{aligned} \log(RSS_k/RSS_1) &= \log\left(1 + \frac{RSS_k - RSS_1}{RSS_1}\right) \\ &= \frac{RSS_k - RSS_1}{RSS_1} + o_p\left(\frac{RSS_k - RSS_1}{RSS_1}\right) \\ &= O_p(C_n^2) + O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{nh^k}\right). \end{aligned}$$

Take  $h_{k,opt} = O(n^{-1/(4+k)})$  for each  $k \geq 1$ . Note that for  $k < 4$ ,  $\sqrt{n} = o(nh_{k,opt}^k)$ . Then we can get, for large  $n$ ,

$$\sqrt{n} \left[ \frac{\log(n)k}{\min(nh_{k,opt}^k, \sqrt{n})} - \frac{\log(n)}{\min(nh_{1,opt}, \sqrt{n})} \right] = \log(n)(k-1).$$

Also note that  $\sqrt{n}C_n^2 = o(1)$  where  $C_n = n^{-1/2}h^{-1/4}$ . Thus for  $k = 2$  and 3, in probability,

$$\sqrt{n}(BIC_k - BIC_1) \rightarrow \infty.$$

For  $k > 4$ ,  $nh_{k,opt}^k = o(\sqrt{n})$ . Then

$$nh_{k,opt}^k \left[ \frac{\log(n)k}{\min(nh_{k,opt}^k, \sqrt{n})} - \frac{\log(n)}{\min(nh_{1,opt}, \sqrt{n})} \right] = nh_{k,opt}^k \left[ \frac{\log(n)k}{nh_{k,opt}^k} - \frac{\log(n)}{\sqrt{n}} \right] \Rightarrow \infty.$$

Note that  $nh_{k,opt}^k C_n^2 = (nh_{k,opt}^{2k})^{1/2} \cdot \sqrt{n}C_n^2 = o(1)$  with  $C_n = n^{-1/2}h^{-1/4}$ . Then in probability we have

$$nh_{k,opt}^k(BIC_k - BIC_1) \rightarrow \infty,$$

for  $k > 4$ . Finally, consider  $k = 4$ . In this situation,  $nh_{k,opt}^k = \sqrt{n}$  and thus

$$\sqrt{n} \left[ \frac{\log(n)k}{\min(nh_{k,opt}^k, \sqrt{n})} - \frac{\log(n)}{\min(nh_{1,opt}, \sqrt{n})} \right] = 3 \log(n).$$

Then, it is easy to see that in probability

$$\sqrt{n}(BIC_4 - BIC_1) \rightarrow \infty.$$

Hence, for large  $n$ ,  $\hat{q} \rightarrow 1$ . We finally conclude that  $P(BIC_k > BIC_1) \rightarrow 1$  for any  $k > 1$ . In other words,  $P(\hat{q} = 1) \rightarrow 1$ .  $\square$

*Proof of Theorem 3.* We first consider the global alternative (3). Under this alternative, Proposition 1 shows that  $\hat{q} \rightarrow q \geq 1$ . According to White (1981),  $\hat{\gamma}$  is a root- $n$  consistent estimator of  $\tilde{\gamma}_0$  which is different from the true value  $\gamma$  under the null hypothesis. Let  $\Delta(\mathbf{x}_i) = G(B^\tau \mathbf{x}_i) - g(\tilde{\beta}_0^\tau \mathbf{x}_i, \tilde{\theta}_0)$ . Then  $\hat{\epsilon}_i = \eta_i + \Delta(\mathbf{x}_i) - (g(\hat{\beta}^\tau \mathbf{x}_i, \hat{\theta}) - g(\tilde{\beta}_0^\tau \mathbf{x}_i, \tilde{\theta}_0))$ . It is then easy to see that  $V_n \Rightarrow E(\Delta^2(\mathbf{X})f(B^\tau \mathbf{X}))$  by using the U-statistics theory. Similarly, we can also have that in probability  $\widehat{Var}$  converges to a positive value which is different from  $Var$ . Then obviously, we can obtain that  $T_n/(nh^{1/2}) \rightarrow Constant > 0$  in probability.

Under the local alternatives (9),  $V_n$  has the following decomposition by

Taylor expansion:

$$\begin{aligned}
V_n &= \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)) (\eta_i + C_n G(B^\tau \mathbf{x}_i)) (\eta_j + C_n G(B^\tau \mathbf{x}_j)) \\
&\quad - \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)) (\eta_i + C_n G(B^\tau \mathbf{x}_i)) m(\mathbf{x}_j, \beta, \theta)^\tau (\hat{\gamma} - \gamma) \\
&\quad + \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)) m(\mathbf{x}_i, \beta, \theta)^\tau (\hat{\gamma} - \gamma) m(\mathbf{x}_j, \beta, \theta)^\tau (\hat{\gamma} - \gamma) \\
&\quad + o_p(\bar{V}_n^*) \\
&=: \bar{V}_{n1} - \bar{V}_{n2} + \bar{V}_{n3} + o_p(\bar{V}_n^*)
\end{aligned} \tag{14}$$

where  $\bar{V}_n^* = \bar{V}_{n1} - \bar{V}_{n2} + \bar{V}_{n3}$ .

For the term  $\bar{V}_{n2}$  in (14), it follows that

$$\begin{aligned}
\bar{V}_{n2} &= \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)) \eta_i m(\mathbf{x}_j, \beta, \theta)^\tau (\hat{\gamma} - \gamma) \\
&\quad + C_n \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau (\mathbf{x}_i - \mathbf{x}_j)) G(B^\tau \mathbf{x}_i) m(\mathbf{x}_j, \beta, \theta)^\tau (\hat{\gamma} - \gamma) \\
&= \bar{V}_{n2,1}(\hat{\gamma} - \gamma) + C_n \bar{V}_{n2,2}^\tau (\hat{\gamma} - \gamma).
\end{aligned}$$

From Lemma 2, we have  $\bar{V}_{n2,1} = O_p(n^{-1/2})$ . It can also be proved that

$$\bar{V}_{n2,2} = E(G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)f(\beta^\tau \mathbf{X})) + o_p(1).$$

When  $C_n = n^{-1/2}h^{-1/4}$ , Lemma 3 implies that

$$\begin{aligned}
&nh^{1/2}\bar{V}_{n2} \\
&= nh^{1/2} \left[ O_p(n^{-1/2})O_p(C_n) \right. \\
&\quad \left. + 2C_n^2 E^\tau (G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)f(\beta^\tau \mathbf{X}))\Sigma_x^{-1} E(G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)) \right] \\
&= 2E^\tau (G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)f(\beta^\tau \mathbf{X}))\Sigma_x^{-1} E(G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)) + o_p(1).
\end{aligned} \tag{15}$$

Now, we turn to consider the term  $\bar{V}_{n3}$ . It is easy to see that

$$\begin{aligned}
\bar{V}_{n3} &= (\hat{\gamma} - \gamma)^\tau E[m(\mathbf{X}, \beta, \theta)m(\mathbf{X}, \beta, \theta)^\tau f(\beta^\tau \mathbf{X})](\hat{\gamma} - \gamma) + o_p(C_n^2) \\
&= C_n^2 E^\tau [G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)]\Sigma_x^{-1} E[m(\mathbf{X}, \beta, \theta)m(\mathbf{X}, \beta, \theta)^\tau f(\beta^\tau \mathbf{X})] \\
&\quad \times \Sigma_x^{-1} E[G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)] + o_p(C_n^2).
\end{aligned}$$



As a result, when  $C_n = n^{-1/2}h^{-1/4}$ , we obtain

$$\begin{aligned} nh^{1/2}\bar{V}_{n3} &= E^\tau[G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)]\Sigma_x^{-1}E[m(\mathbf{X}, \beta, \theta)m(\mathbf{X}, \beta, \theta)^\tau f(\beta^\tau \mathbf{X})] \\ &\quad \times \Sigma_x^{-1}E[G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)] + o_p(1). \end{aligned} \quad (16)$$

Now we investigate the term  $\bar{V}_{n1}$  in (14), it can be decomposed as:

$$\begin{aligned} \bar{V}_{n1} &= \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau(\mathbf{x}_i - \mathbf{x}_j))\eta_i\eta_j \\ &\quad + C_n \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau(\mathbf{x}_i - \mathbf{x}_j))\eta_i G(B^\tau \mathbf{x}_j) \\ &\quad + C_n^2 \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n K_h(\hat{B}(\hat{q})^\tau(\mathbf{x}_i - \mathbf{x}_j))G(B^\tau \mathbf{x}_i)G(B^\tau \mathbf{x}_j) \\ &= \bar{V}_{n1,1} + C_n \bar{V}_{n1,2} + C_n^2 \bar{V}_{n1,3}. \end{aligned}$$

From the proof of Theorem 1 and the conclusion of Lemma 2, we have

$$\begin{aligned} \bar{V}_{n1,2} &= O_p(n^{-1/2}), \\ \bar{V}_{n1,3} &= E(G^2(B^\tau \mathbf{X})f(\beta^\tau \mathbf{X})) + o_p(1). \end{aligned}$$

Note that  $nh^{1/2}\bar{V}_{n1,1} \Rightarrow N(0, Var)$ . Consequently, when  $C_n = n^{-1/2}h^{-1/4}$

$$nh^{1/2}\bar{V}_{n1} \Rightarrow N(E(G^2(B^\tau \mathbf{X})f(\beta^\tau \mathbf{X})), Var). \quad (17)$$

Combining equations (15), (16) and (17), we can have

$$nh^{1/2}V_n \Rightarrow N(\mu, Var),$$

where

$$\begin{aligned} \mu &= E[G^2(B^\tau \mathbf{X})f(\beta^\tau \mathbf{X})] - 2E^\tau[H(\mathbf{X})f(\beta^\tau \mathbf{X})]\Sigma_x^{-1}E[H(\mathbf{X})] \\ &\quad + E^\tau[H(\mathbf{X})]\Sigma_x^{-1}E[m(\mathbf{X}, \beta, \theta)m(\mathbf{X}, \beta, \theta)^\tau f(\beta^\tau \mathbf{X})]\Sigma_x^{-1}E[H(\mathbf{X})] \\ &= E\left[\left(G(B^\tau \mathbf{X}) - m(\mathbf{X}, \beta, \theta)^\tau \Sigma_x^{-1}E[H(\mathbf{X})]\right)^2 f(\beta^\tau \mathbf{X})\right]. \end{aligned}$$

here  $H(\mathbf{X}) = G(B^\tau \mathbf{X})m(\mathbf{X}, \beta, \theta)$ .

Let  $T_n$  be the standardized version of  $V_n$  of the form:

$$T_n = \frac{h^{(1-\hat{q})/2} \sum_{i=1}^n \sum_{j \neq i}^n \hat{\epsilon}_i \hat{\epsilon}_j K\left(\frac{\hat{B}(\hat{q})^\tau(\mathbf{x}_i - \mathbf{x}_j)}{h}\right)}{\{2 \sum_{i=1}^n \sum_{j \neq i}^n K^2\left(\frac{\hat{B}(\hat{q})^\tau(\mathbf{x}_i - \mathbf{x}_j)}{h}\right) \hat{\epsilon}_i^2 \hat{\epsilon}_j^2\}^{1/2}}.$$

Note that when  $C_n = n^{-1/2}h^{-1/4}$ ,  $\widehat{Var}$  is still a consistent estimate of  $Var$ . From Lemma 3, we can easily have that  $\hat{\beta}$  is also consistent under the local alternative. Thus both  $\hat{\beta}$  and  $\widehat{Var}$  are still consistent to  $\beta$  and  $Var$  under the local alternatives. Thus  $T_n^2 \Rightarrow \chi_1^2(\mu^2/Var)$ .

When  $C_n$  has a slower convergence rate than  $n^{-1/2}h^{-1/4}$ , the above arguments can show that the test statistic goes to infinity in probability. The details are omitted. Theorem 3 is proved.  $\square$

## References

- [1] Cook, R. D. (1998). *Regression Graphics: Ideas for Studying Regressions Through Graphics*. New York: Wiley.
- [2] Cook, R. D. and Forzani, L. (2009). Likelihood-based sufficient dimension reduction. *J. Amer. Statist. Assoc.*, **104**, 197-208.
- [3] Cook, R. D. and Li, B. (2002). Dimension reduction for conditional mean in regression. *Ann. of Statist.*, **30**, 455-474.
- [4] Cook, R. D. and Weisberg, S. (1991). Discussion of “Sliced inverse regression for dimension reduction,” by K. C. Li, *J. Amer. Statist. Assoc.*, **86**, 316-342.
- [5] Delgado, M. A. and Gonz  les-Manteiga, W. (2001). Significance testing in nonparametric regression based on the bootstrap, *Ann. of Statist.*, **29**, 1469-1507.
- [6] Dette, H. (1999). A consistent test for the functional form of a regression based on a difference of variance estimates. *Ann. of Statist.*, **27**, 1012-1050.
- [7] Dette, H., Neumeyer, N. and Van Keilegom, I. (2007). A new test for the parametric form of the variance function in nonparametric regression. *J. Roy. Statist. Soc. Ser. B.*, **69**, 903-917.
- [8] Fan, J., Zhang, C. and Zhang, J. (2001) Generalized likelihood ratio statistics and Wilks phenomenon. *Ann. of Statist.*, **29**, 153-193.
- [9] Fan, Y. and Li, Q., (1996). Consistent model specification tests: omitted variables and semiparametric functional forms. *Econometrica*, **64**, 865-890.

- [10] González-Manteiga, W. and Crujeiras, R. M. (2013). An updated review of Goodness-of-Fit tests for regression models. *Test*, **22**, 361-411.
- [11] Härdle, W., Huet, S., Mammen, E. and Sperlich, S. (2004). Bootstrap inference in semiparametric generalized additive models. *Econometric Theory*, **20**, 265-300.
- [12] Härdle, W. and Mammen, E. (1993). Comparing nonparametric versus parametric regression fits. *Ann. of Statist.*, **21**, 1926-1947.
- [13] Jennrich, R. I. (1969). Asymptotic properties of non-least squares estimators, *Ann. Math. Stat.*, **40**, 633-643.
- [14] Khmadladze, E. V. and Koul, H. L. (2004). Martingale transforms goodness-of-fit tests in regression models. *Ann. of Statist.*, **37**, 995-1034.
- [15] Koul, H. L. and Ni, P. P. (2004). Minimum distance regression model checking. *J. Stat. Plan. Infer.*, **119**, 109-141.
- [16] Li, B. and Wang, S. (2007). On directional regression for dimension reduction. *J. Amer. Statist. Assoc.*, **102**, 997-1008.
- [17] Li, B., Wen, S. Q. and Zhu, L. X. (2008). On a Projective Resampling method for dimension reduction with multivariate responses. *J. Amer. Statist. Assoc.*, **103**, 1177-1186.
- [18] Li, B., Zha, H. and Chiaromonte, F. (2005). Contour Regression: A General Approach to Dimension Reduction, *Ann. of Statist.*, **33**, 1580-1616.
- [19] Li, K. C. (1991). Sliced inverse regression for dimension reduction, *J. Amer. Statist. Assoc.*, **86**, 86, 316-327.
- [20] Li, Y. X. and Zhu, L. X. (2007). Asymptotics for sliced average variance estimation, *Ann. of Statist.*, **35**, 41-69.
- [21] Little, R. J. A. and Rubin, D. B. (1987). *Statistical Analysis with Missing Data*. Wiley, New York.
- [22] Mora, J. and Moro-Egido, A. I. (2008). On specification testing of ordered discrete choice models, *Journal of Economics*, **143**, 191 - 205.

- [23] Neumeyer, N. and Van Keilegom, I. (2010). Estimating the error distribution in nonparametric multiple regression with applications to model testing. *J. Multivariate Anal.*, **101**, 1067-1078.
- [24] Quinlan, R. (1993). Combining Instance-Based and Model-Based Learning. *In Proceedings on the Tenth International Conference of Machine Learning*, 236-243, University of Massachusetts, Amherst. Morgan Kaufmann.
- [25] Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. John Wiley, New York.
- [26] Stute, W. (1997). Nonparametric model checks for regression. *Ann. of Statist.*, **25**, 613-641.
- [27] Stute, W., Gonzáles-Manteiga, W. and Presedo-Quindimil, M. (1998a). Bootstrap approximation in model checks for regression. *J. Amer. Statist. Assoc.*, **93**, 141-149.
- [28] Stute, W., Thies, S. and Zhu, L. X. (1998b). Model checks for regression: An innovation process approach. *Ann. of Statist.*, **26**, 1916-1934.
- [29] Stute, W. and Zhu, L. X. (2002). Model checks for generalized linear models. *Scand. J. Stat.*, **29**, 535-546.
- [30] Van Keilegom, I., Gonzáles-Manteiga, W. and Sánchez Sellero, C. (2008). Goodness-of-fit tests in parametric regression based on the estimation of the error distribution. *Test*, **17**, 401-415.
- [31] Wang, Q. and Yin, X. R. (2008). A nonlinear multi-dimensional variable selection method for high dimensional data: Sparse MAVE. *Comput. Stat. Data Anal.*, **52**, 4512-4520.
- [32] White, H. (1981). Consequences and detection of misspecified nonlinear regression models. *J. Am. Statist. Assoc.*, **76**, 419-433.
- [33] Wu, C. F. (1986). Jackknife, bootstrap and other resampling methods in regression analysis. *Ann. of Statist.*, **14**, 1261-1295.
- [34] Xia, Y. C. (2006). Asymptotic distributions for two estimators of the single-index model. *Econometric Theory*, **22**, 1112-1137.
- [35] Xia, Y. (2007). A constructive approach to the estimation of dimension reduction directions. *Ann. of Statist.*, **35**, 2654-2690.

- [36] Xia, Y. (2009). Model check for multiple regressions via dimension reduction *Biometrika*, **96**, 133-148.
- [37] Xia, Y. C., Tong, H., Li, W. K. and Zhu, L. X. (2002). An adaptive estimation of dimension reduction space. *J. Roy. Statist. Soc. Ser. B.*, **64**, 363-410.
- [38] Zhang, C. and Dette, H. (2004). A power comparison between nonparametric regression tests. *Stat. Probab. Lett.*, **66**, 289-301.
- [39] Zheng, J. X. (1996). A consistent test of functional form via nonparametric estimation techniques. *J. Econometrics.*, **75**, 263-289.
- [40] Zhu, L. P., Wang, T., Zhu, L.X. and Ferré, L. (2010a). Sufficient dimension reduction through discretization-expectation estimation. *Biometrika*, **97**. 295-304.
- [41] Zhu, L. P., Zhu, L. X. and Feng, Z. H. (2010b). Dimension reduction in regressions through cumulative slicing estimation. *J. Am. Statist. Assoc.* , **105**, 1455-1466.
- [42] Zhu, L. X., Miao, B. Q. and Peng, H. (2006). Sliced inverse regression with large dimensional covariates. *J. Am. Statist. Assoc.*, **101**, 630-643.
- [43] Zhu, L. X. and Ng, K. (1995). Asymptotics for Sliced Inverse Regression. *Statis. Sini.*, **5**, 727- 736.

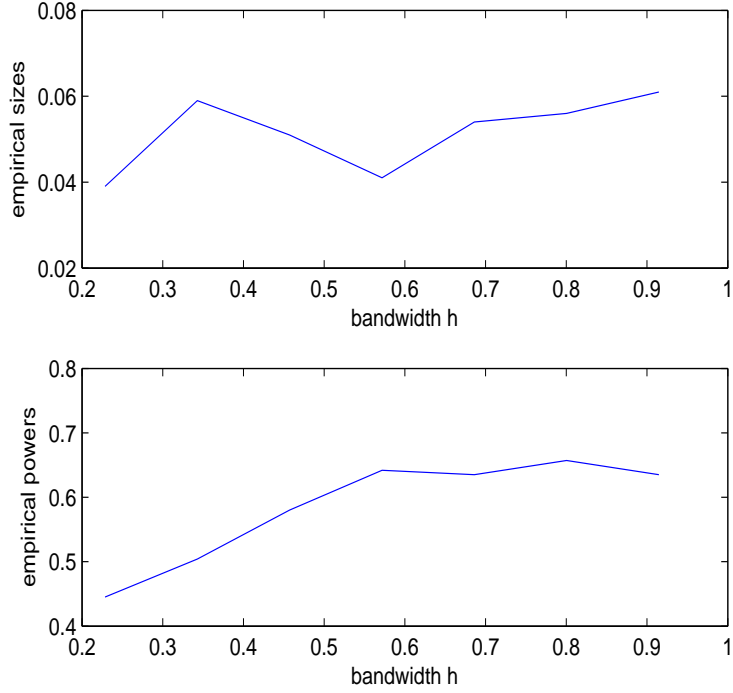


Figure 1: The empirical size and power curves of  $T_n^{DEE}$  against the bandwidth  $h$  with  $X \sim N(0, \Sigma_1)$ ,  $\epsilon \sim N(0, 1)$  and sample size 50 under different choices of  $a$  for study 1 with  $a = 0$ (the above panel) and  $a = 1$ (the below panel).

Table 1: Empirical sizes and powers of  $\tilde{T}_n^{MAVE}$  and  $T_n^{DEE}$  for  $H_0$  vs.  $H_{11}$  and  $H_{12}$ , with  $X \sim N(0, \Sigma_i)$ ,  $i = 1, 2$  and  $\epsilon \sim N(0, 1)$ .

|                                 |     | $\tilde{T}_n^{MAVE}$ |           | $T_n^{DEE}$ |           |
|---------------------------------|-----|----------------------|-----------|-------------|-----------|
| $a$                             |     | $n = 50$             | $n = 100$ | $n = 50$    | $n = 100$ |
| $H_{11}, X \sim N(0, \Sigma_1)$ | 0   | 0.0630               | 0.0565    | 0.0470      | 0.0500    |
|                                 | 0.2 | 0.0890               | 0.1535    | 0.0730      | 0.1263    |
|                                 | 0.4 | 0.2175               | 0.4735    | 0.1623      | 0.3857    |
|                                 | 0.6 | 0.4290               | 0.8125    | 0.3207      | 0.7227    |
|                                 | 0.8 | 0.6470               | 0.9630    | 0.4910      | 0.9207    |
|                                 | 1.0 | 0.8135               | 0.9990    | 0.6347      | 0.9803    |
| $X \sim N(0, \Sigma_2)$         | 0   | 0.0460               | 0.0545    | 0.0480      | 0.0563    |
|                                 | 0.2 | 0.0570               | 0.1050    | 0.0767      | 0.1173    |
|                                 | 0.4 | 0.1165               | 0.3255    | 0.1647      | 0.3667    |
|                                 | 0.6 | 0.2275               | 0.6530    | 0.3243      | 0.7203    |
|                                 | 0.8 | 0.4100               | 0.8885    | 0.4953      | 0.9293    |
|                                 | 1.0 | 0.5230               | 0.9690    | 0.6787      | 0.9887    |
| $H_{12}, X \sim N(0, \Sigma_1)$ | 0   | 0.0535               | 0.0610    | 0.0490      | 0.0470    |
|                                 | 0.2 | 0.1275               | 0.1845    | 0.0990      | 0.1687    |
|                                 | 0.4 | 0.2950               | 0.5695    | 0.2657      | 0.5510    |
|                                 | 0.6 | 0.5715               | 0.8895    | 0.5383      | 0.8980    |
|                                 | 0.8 | 0.8100               | 0.9945    | 0.7763      | 0.9890    |
|                                 | 1.0 | 0.9410               | 1.0000    | 0.9267      | 0.9993    |
| $X \sim N(0, \Sigma_2)$         | 0   | 0.0395               | 0.0380    | 0.0460      | 0.0523    |
|                                 | 0.2 | 0.0700               | 0.1160    | 0.0773      | 0.1140    |
|                                 | 0.4 | 0.1545               | 0.3690    | 0.1820      | 0.3717    |
|                                 | 0.6 | 0.3285               | 0.6920    | 0.3660      | 0.6953    |
|                                 | 0.8 | 0.5490               | 0.9025    | 0.5723      | 0.9130    |
|                                 | 1.0 | 0.7340               | 0.9805    | 0.7587      | 0.9857    |

Table 2: Empirical sizes and powers of  $T_n^{MAVE*}$  and  $T_n^{DEE*}$  for  $H_0$  vs.  $H_{11}$  and  $H_{12}$ , with  $X \sim N(0, \Sigma_i)$ ,  $i = 1, 2$  and  $\epsilon \sim N(0, 1)$ .

|                                 |     | $T_n^{MAVE*}$ |           | $T_n^{DEE*}$ |           |
|---------------------------------|-----|---------------|-----------|--------------|-----------|
| $a$                             |     | $n = 50$      | $n = 100$ | $n = 50$     | $n = 100$ |
| $H_{11}, X \sim N(0, \Sigma_1)$ | 0   | 0.0697        | 0.0580    | 0.0470       | 0.0500    |
|                                 | 0.2 | 0.1160        | 0.1890    | 0.0840       | 0.1425    |
|                                 | 0.4 | 0.2740        | 0.5260    | 0.1635       | 0.3900    |
|                                 | 0.6 | 0.4670        | 0.8510    | 0.3255       | 0.7235    |
|                                 | 0.8 | 0.6750        | 0.9750    | 0.5115       | 0.9185    |
|                                 | 1.0 | 0.8320        | 0.9960    | 0.6160       | 0.9780    |
| $X \sim N(0, \Sigma_2)$         | 0   | 0.0490        | 0.0550    | 0.0500       | 0.0475    |
|                                 | 0.2 | 0.0770        | 0.1160    | 0.0680       | 0.1285    |
|                                 | 0.4 | 0.1600        | 0.3460    | 0.1515       | 0.3480    |
|                                 | 0.6 | 0.2850        | 0.7170    | 0.3080       | 0.7155    |
|                                 | 0.8 | 0.4520        | 0.8930    | 0.4835       | 0.9255    |
|                                 | 1.0 | 0.5800        | 0.9740    | 0.6660       | 0.9820    |
| $H_{12}, X \sim N(0, \Sigma_1)$ | 0   | 0.0770        | 0.0740    | 0.0435       | 0.0555    |
|                                 | 0.2 | 0.1520        | 0.2070    | 0.1095       | 0.1680    |
|                                 | 0.4 | 0.3320        | 0.5990    | 0.2545       | 0.5510    |
|                                 | 0.6 | 0.6170        | 0.9200    | 0.5600       | 0.8975    |
|                                 | 0.8 | 0.8410        | 0.9990    | 0.7690       | 0.9925    |
|                                 | 1.0 | 0.9430        | 1.0000    | 0.9215       | 1.0000    |
| $X \sim N(0, \Sigma_2)$         | 0   | 0.0570        | 0.058     | 0.0420       | 0.0540    |
|                                 | 0.2 | 0.0990        | 0.1250    | 0.0705       | 0.1275    |
|                                 | 0.4 | 0.2070        | 0.3560    | 0.1770       | 0.3495    |
|                                 | 0.6 | 0.3710        | 0.7070    | 0.3380       | 0.6870    |
|                                 | 0.8 | 0.5920        | 0.9150    | 0.5390       | 0.9190    |
|                                 | 1.0 | 0.7600        | 0.9810    | 0.7445       | 0.9805    |



Table 3: Empirical sizes and powers for  $H_0$  vs.  $H_{13}$ , with  $X \sim N(0, \Sigma_i)$ ,  $i = 1, 2$  and  $\epsilon \sim N(0, 1)$ .

|  | $a$ | $\tilde{T}_n^{MAVE}$ |           | $T_n^{DEE}$  |           |
|--|-----|----------------------|-----------|--------------|-----------|
|  |     | $n = 50$             | $n = 100$ | $n = 50$     | $n = 100$ |
| $X \sim N(0, \Sigma_1), \epsilon \sim N(0, 1)$ | 0   | 0.0515               | 0.0625    | 0.0507       | 0.0563    |
|  | 0.2 | 0.1255               | 0.2245    | 0.1067       | 0.2000    |
|  | 0.4 | 0.3465               | 0.7520    | 0.3330       | 0.7127    |
|  | 0.6 | 0.6390               | 0.9790    | 0.6170       | 0.9580    |
|  | 0.8 | 0.8335               | 0.9980    | 0.8023       | 0.9960    |
|  | 1.0 | 0.9240               | 1.0000    | 0.8897       | 0.9993    |
| $X \sim N(0, \Sigma_2), \epsilon \sim N(0, 1)$ | 0   | 0.0485               | 0.0505    | 0.0477       | 0.0480    |
|  | 0.2 | 0.4160               | 0.8745    | 0.4163       | 0.8523    |
|  | 0.4 | 0.8760               | 0.9995    | 0.8933       | 0.9993    |
|  | 0.6 | 0.9515               | 1.0000    | 0.9680       | 0.9997    |
|  | 0.8 | 0.9750               | 1.0000    | 0.9860       | 1.0000    |
|  | 1.0 | 0.9765               | 1.0000    | 0.9907       | 1.0000    |
|  | $a$ | $T_n^{MAVE*}$        |           | $T_n^{DEE*}$ |           |
|  |     | $n = 50$             | $n = 100$ | $n = 50$     | $n = 100$ |
| $X \sim N(0, \Sigma_1), \epsilon \sim N(0, 1)$ | 0   | 0.0767               | 0.0640    | 0.0490       | 0.0490    |
|  | 0.2 | 0.1580               | 0.2570    | 0.1075       | 0.1835    |
|  | 0.4 | 0.3860               | 0.7190    | 0.3120       | 0.6800    |
|  | 0.6 | 0.6170               | 0.9560    | 0.5655       | 0.9410    |
|  | 0.8 | 0.7940               | 0.9920    | 0.7400       | 0.9885    |
|  | 1.0 | 0.8660               | 0.9930    | 0.8475       | 0.9930    |
| $X \sim N(0, \Sigma_2), \epsilon \sim N(0, 1)$ | 0   | 0.0580               | 0.0435    | 0.0400       | 0.0475    |
|  | 0.2 | 0.4170               | 0.8310    | 0.3925       | 0.7855    |
|  | 0.4 | 0.7950               | 0.9880    | 0.7755       | 0.9870    |
|  | 0.6 | 0.8770               | 0.9920    | 0.8985       | 0.9955    |
|  | 0.8 | 0.8870               | 1.0000    | 0.9390       | 1.0000    |
|  | 1.0 | 0.8880               | 1.0000    | 0.9445       | 1.0000    |

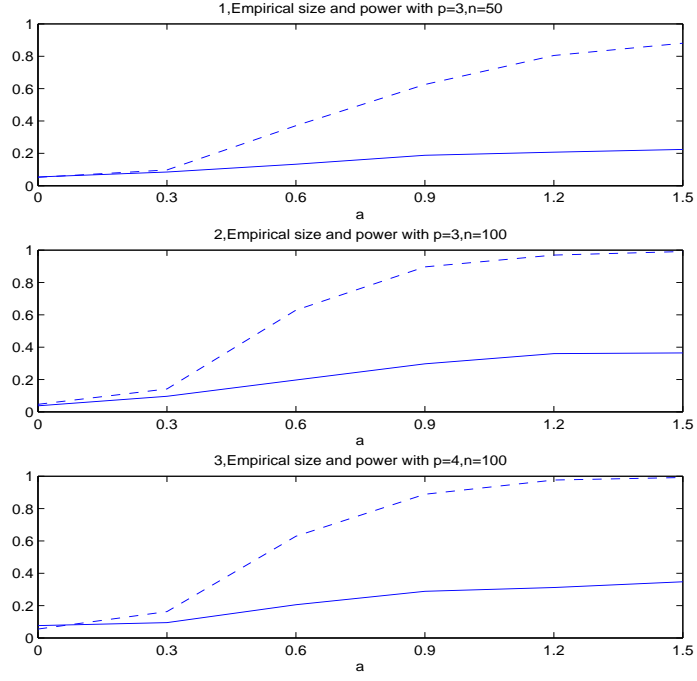


Figure 2: The empirical size and power curves of  $T_n^{SZ}$  and  $T_n^{DEE}$  in Study 2. The solid and dash line represent the results from  $T_n^{SZ}$  and  $T_n^{DEE}$  respectively.

Table 4: Empirical sizes and powers in *Study 3*, with  $p = 2$ . Here cases 1-4 respectively represent situations with  $X \sim N(0, \Sigma_1)$ ,  $\epsilon \sim N(0, 1)$ ; (case 1) or  $DE(0, \sqrt{2}/2)$  (case 2) and  $X \sim N(0, \Sigma_2)$ ,  $\epsilon \sim N(0, 1)$  (case 3) or  $DE(0, \sqrt{2}/2)$  (case 4) respectively.

|       | $a$ | $T_n^{ZH}$ |           | $T_n^{ZH*}$ |           | $T_n^{DEE}$ |           | $T_n^{DEE*}$ |           |
|-------|-----|------------|-----------|-------------|-----------|-------------|-----------|--------------|-----------|
|       |     | $n = 50$   | $n = 100$ | $n = 50$    | $n = 100$ | $n = 50$    | $n = 100$ | $n = 50$     | $n = 100$ |
| Case1 | 0   | 0.0470     | 0.0375    | 0.0500      | 0.0490    | 0.0483      | 0.0453    | 0.0463       | 0.0527    |
|       | 0.2 | 0.0820     | 0.1390    | 0.0905      | 0.1335    | 0.1523      | 0.2653    | 0.1390       | 0.2683    |
|       | 0.4 | 0.2430     | 0.5065    | 0.2560      | 0.4885    | 0.4127      | 0.7610    | 0.4053       | 0.7530    |
|       | 0.6 | 0.5175     | 0.8635    | 0.4245      | 0.8320    | 0.6747      | 0.9650    | 0.6573       | 0.9557    |
|       | 0.8 | 0.7335     | 0.9825    | 0.6350      | 0.9465    | 0.8453      | 0.9943    | 0.8237       | 0.9933    |
|       | 1.0 | 0.8875     | 0.9990    | 0.7175      | 0.9750    | 0.9227      | 1.0000    | 0.9010       | 1.0000    |
| Case2 | 0   | 0.0410     | 0.0375    | 0.0370      | 0.0585    | 0.0497      | 0.0487    | 0.0453       | 0.0530    |
|       | 0.2 | 0.0915     | 0.1260    | 0.1005      | 0.1485    | 0.1653      | 0.2867    | 0.1540       | 0.2867    |
|       | 0.4 | 0.2725     | 0.5300    | 0.2630      | 0.5035    | 0.4293      | 0.7540    | 0.4213       | 0.7683    |
|       | 0.6 | 0.5460     | 0.8635    | 0.4840      | 0.8335    | 0.6757      | 0.9593    | 0.6643       | 0.9557    |
|       | 0.8 | 0.7565     | 0.9790    | 0.6115      | 0.9520    | 0.8370      | 0.9927    | 0.8287       | 0.9947    |
|       | 1.0 | 0.8745     | 0.9990    | 0.7220      | 0.9675    | 0.9183      | 0.9978    | 0.9000       | 0.9965    |
| Case3 | 0   | 0.0370     | 0.0405    | 0.0435      | 0.0590    | 0.0473      | 0.0527    | 0.0597       | 0.0513    |
|       | 0.2 | 0.0830     | 0.1130    | 0.0945      | 0.1760    | 0.1237      | 0.2107    | 0.1087       | 0.2130    |
|       | 0.4 | 0.2630     | 0.5340    | 0.2705      | 0.5220    | 0.3667      | 0.6417    | 0.3320       | 0.6410    |
|       | 0.6 | 0.5295     | 0.8955    | 0.4910      | 0.8505    | 0.6220      | 0.9363    | 0.5913       | 0.9250    |
|       | 0.8 | 0.7915     | 0.9855    | 0.6385      | 0.9580    | 0.8170      | 0.9923    | 0.7797       | 0.9870    |
|       | 1.0 | 0.9115     | 0.9995    | 0.7300      | 0.9775    | 0.9183      | 0.9990    | 0.8757       | 0.9987    |
| Case4 | 0   | 0.0370     | 0.0410    | 0.0455      | 0.0420    | 0.0463      | 0.0513    | 0.0565       | 0.0565    |
|       | 0.2 | 0.0890     | 0.1495    | 0.0995      | 0.1540    | 0.1367      | 0.2140    | 0.1305       | 0.2105    |
|       | 0.4 | 0.2960     | 0.5490    | 0.2680      | 0.5285    | 0.3677      | 0.6623    | 0.3690       | 0.6695    |
|       | 0.6 | 0.5750     | 0.8955    | 0.5060      | 0.8500    | 0.6450      | 0.9333    | 0.5960       | 0.9175    |
|       | 0.8 | 0.7885     | 0.9895    | 0.6600      | 0.9510    | 0.8147      | 0.9897    | 0.7725       | 0.9825    |
|       | 1.0 | 0.9005     | 0.9980    | 0.7380      | 0.9790    | 0.9110      | 0.9997    | 0.8855       | 0.9980    |

Table 5: Empirical sizes and powers in *Study 3*, with  $p = 8$ . Here cases 1-4 represent the situations with  $X \sim N(0, \Sigma_1)$ ,  $\epsilon \sim N(0, 1)$  (case 1) or  $DE(0, \sqrt{2}/2)$  (case 2) and  $X \sim N(0, \Sigma_2)$ ,  $\epsilon \sim N(0, 1)$  (case 3) or  $DE(0, \sqrt{2}/2)$  (case 4) respectively.

|       | $a$ | $T_n^{ZH}$ |           | $T_n^{ZH*}$ |           | $T_n^{DEE}$ |           | $T_n^{DEE*}$ |           |
|-------|-----|------------|-----------|-------------|-----------|-------------|-----------|--------------|-----------|
|       |     | $n = 50$   | $n = 100$ | $n = 50$    | $n = 100$ | $n = 50$    | $n = 100$ | $n = 50$     | $n = 100$ |
| Case1 | 0   | 0.0182     | 0.0297    | 0.0450      | 0.0415    | 0.0605      | 0.0460    | 0.0465       | 0.0495    |
|       | 0.2 | 0.0280     | 0.0475    | 0.0500      | 0.0705    | 0.1400      | 0.2645    | 0.1345       | 0.2610    |
|       | 0.4 | 0.0442     | 0.0795    | 0.0785      | 0.0930    | 0.3485      | 0.6990    | 0.3555       | 0.7145    |
|       | 0.6 | 0.0742     | 0.1573    | 0.1035      | 0.1895    | 0.5905      | 0.9420    | 0.5555       | 0.9280    |
|       | 0.8 | 0.1022     | 0.2627    | 0.1330      | 0.2770    | 0.7510      | 0.9855    | 0.7275       | 0.9890    |
|       | 1.0 | 0.1422     | 0.3715    | 0.1630      | 0.3500    | 0.8475      | 0.9960    | 0.8170       | 0.9935    |
| Case2 | 0   | 0.0175     | 0.0265    | 0.0480      | 0.0430    | 0.0543      | 0.0517    | 0.0400       | 0.0540    |
|       | 0.2 | 0.0283     | 0.0508    | 0.0655      | 0.0590    | 0.1463      | 0.2843    | 0.1395       | 0.2760    |
|       | 0.4 | 0.0522     | 0.0953    | 0.0865      | 0.1240    | 0.3740      | 0.7240    | 0.3610       | 0.7365    |
|       | 0.6 | 0.0885     | 0.1955    | 0.1295      | 0.2070    | 0.6073      | 0.9323    | 0.5990       | 0.9260    |
|       | 0.8 | 0.1323     | 0.2953    | 0.1560      | 0.2925    | 0.7470      | 0.9873    | 0.7355       | 0.9860    |
|       | 1.0 | 0.1675     | 0.4070    | 0.1880      | 0.3955    | 0.8510      | 0.9980    | 0.8170       | 0.9935    |
| Case3 | 0   | 0.0213     | 0.0280    | 0.0480      | 0.0485    | 0.0463      | 0.0483    | 0.0450       | 0.0590    |
|       | 0.2 | 0.0600     | 0.1335    | 0.0765      | 0.1660    | 0.3237      | 0.6443    | 0.2905       | 0.6595    |
|       | 0.4 | 0.1935     | 0.4572    | 0.2245      | 0.4265    | 0.6970      | 0.9797    | 0.6780       | 0.9780    |
|       | 0.6 | 0.3445     | 0.7280    | 0.3220      | 0.6570    | 0.8773      | 0.9993    | 0.8340       | 0.9980    |
|       | 0.8 | 0.4758     | 0.8588    | 0.4290      | 0.7535    | 0.9237      | 0.9993    | 0.8985       | 0.9990    |
|       | 1.0 | 0.5480     | 0.9205    | 0.4750      | 0.8020    | 0.9527      | 1.0000    | 0.9335       | 0.9990    |
| Case4 | 0   | 0.0190     | 0.0275    | 0.0460      | 0.0540    | 0.0507      | 0.0533    | 0.0475       | 0.0500    |
|       | 0.2 | 0.0742     | 0.1465    | 0.1095      | 0.1915    | 0.3280      | 0.6583    | 0.3140       | 0.6600    |
|       | 0.4 | 0.2350     | 0.4950    | 0.2370      | 0.4420    | 0.6977      | 0.9783    | 0.6975       | 0.9760    |
|       | 0.6 | 0.3757     | 0.7445    | 0.3440      | 0.6495    | 0.8660      | 0.9993    | 0.8355       | 0.9985    |
|       | 0.8 | 0.4765     | 0.8585    | 0.4285      | 0.7425    | 0.9250      | 0.9997    | 0.9110       | 1.0000    |
|       | 1.0 | 0.5537     | 0.9147    | 0.4865      | 0.8095    | 0.9550      | 1.0000    | 0.9355       | 1.0000    |

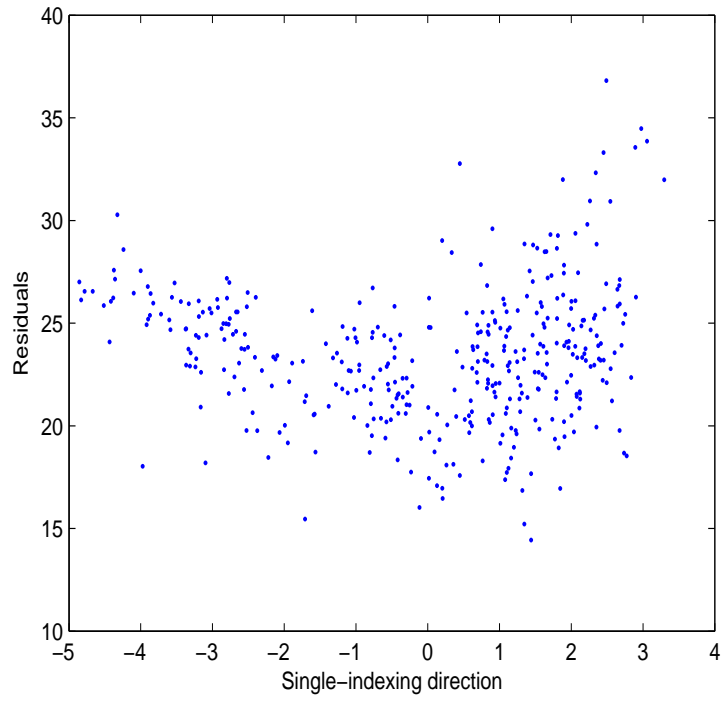


Figure 3: Plot of the residuals from the linear regression model against the single-indexing direction obtained from DEE in the real data analysis.